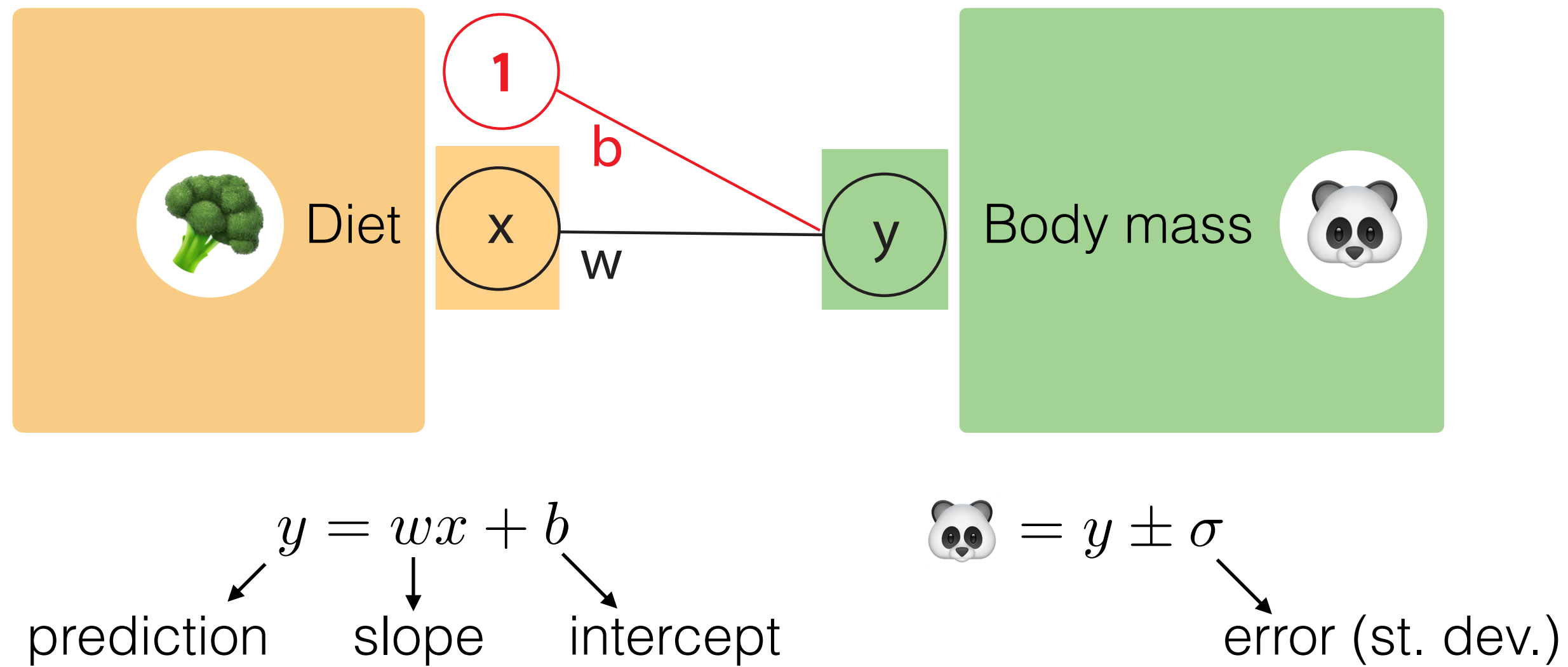
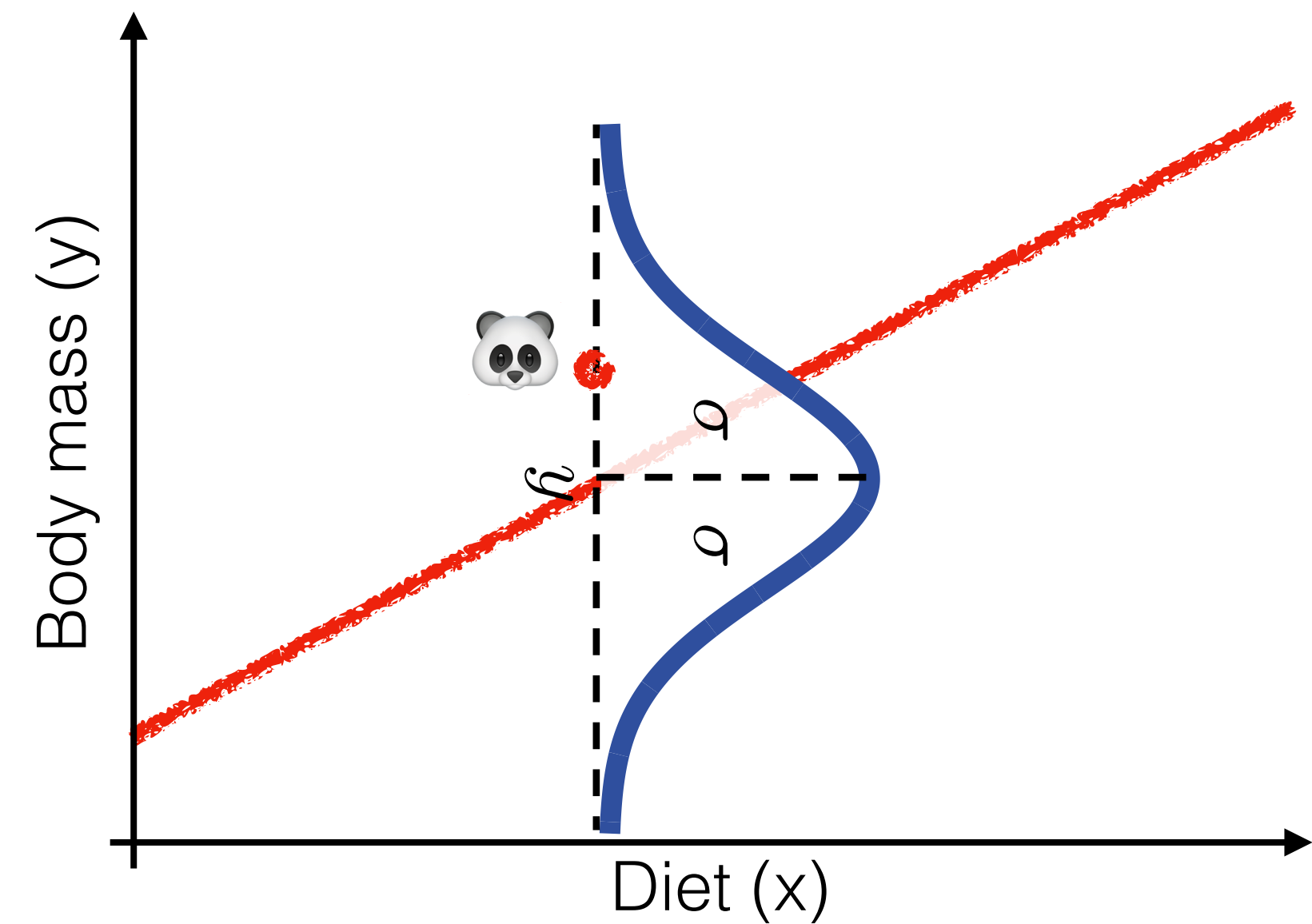


Linear regression



Likelihood of an observed panda body mass

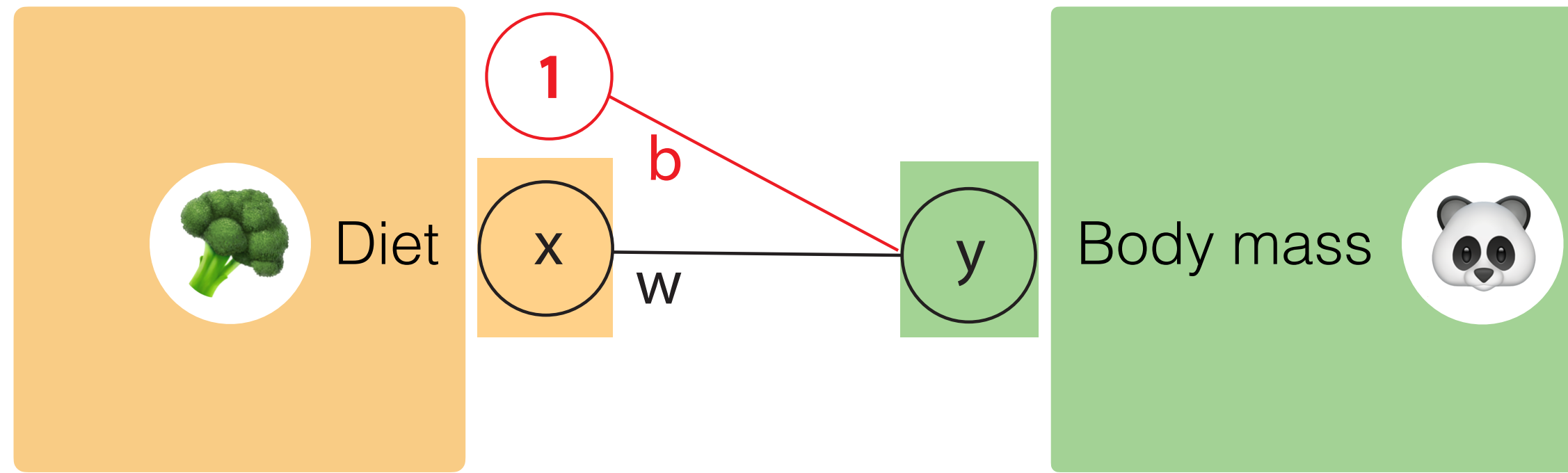
$$P(\text{panda} | x, w, b, \sigma) \sim \mathcal{N}(y, \sigma)$$



Probability density function of a normal distribution

$$P(\text{panda} | y, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{panda} - y)^2}{2\sigma^2}}$$

Linear regression



$$y = wx + b$$

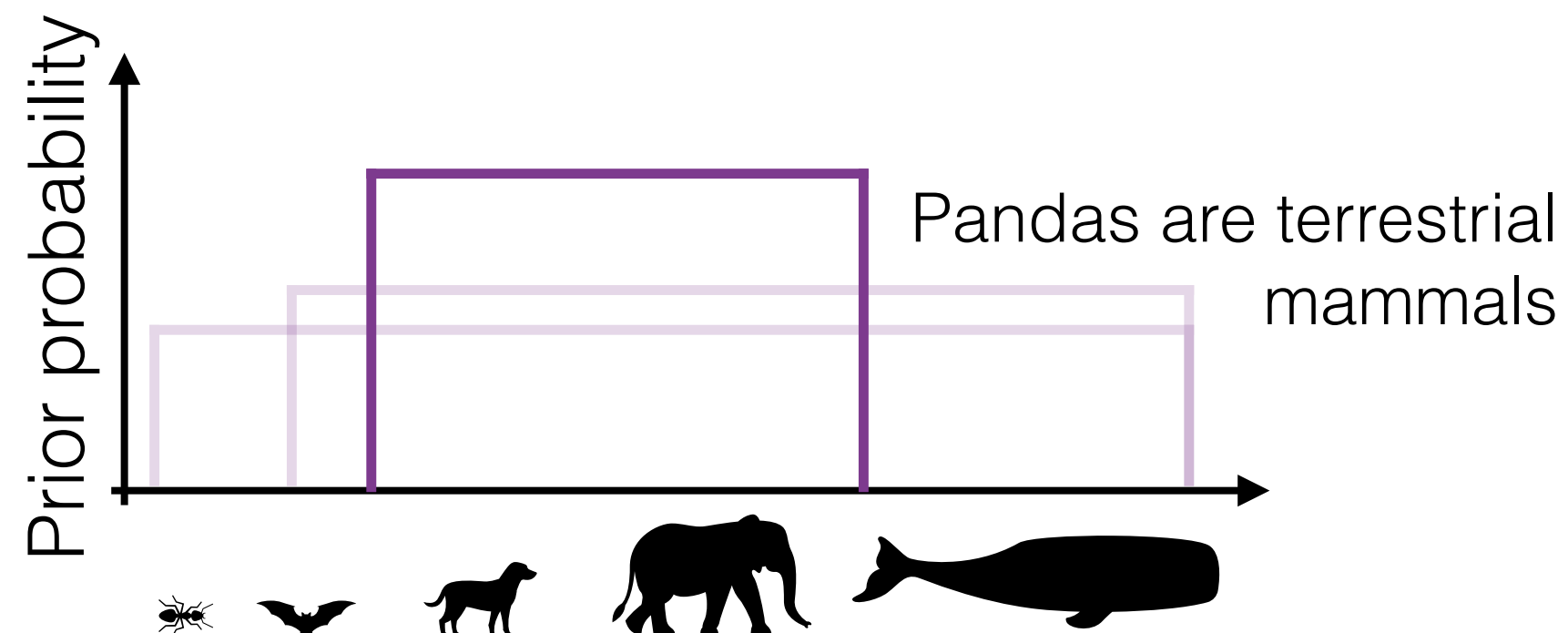
prediction slope intercept

$$\text{panda} = y \pm \sigma$$

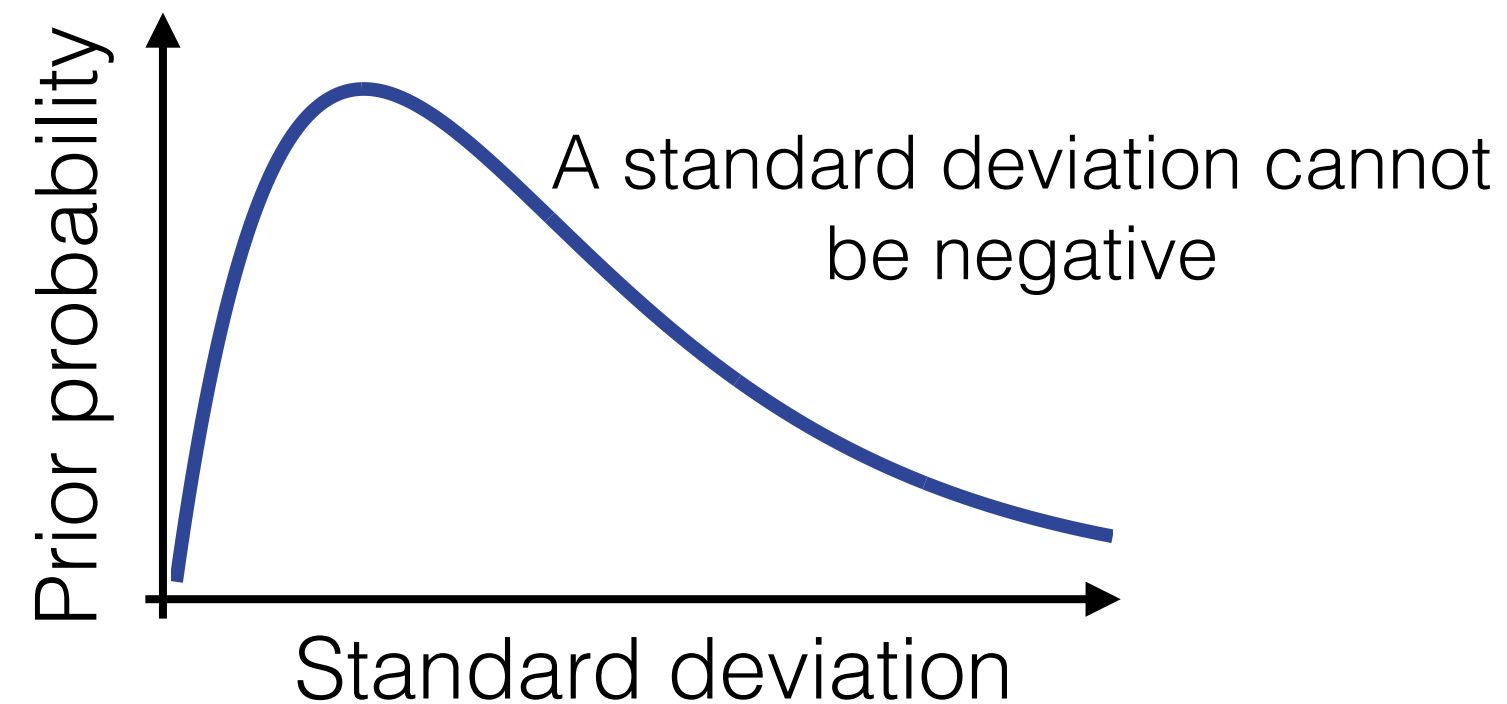
error (st. dev.)

Priors on the model parameters

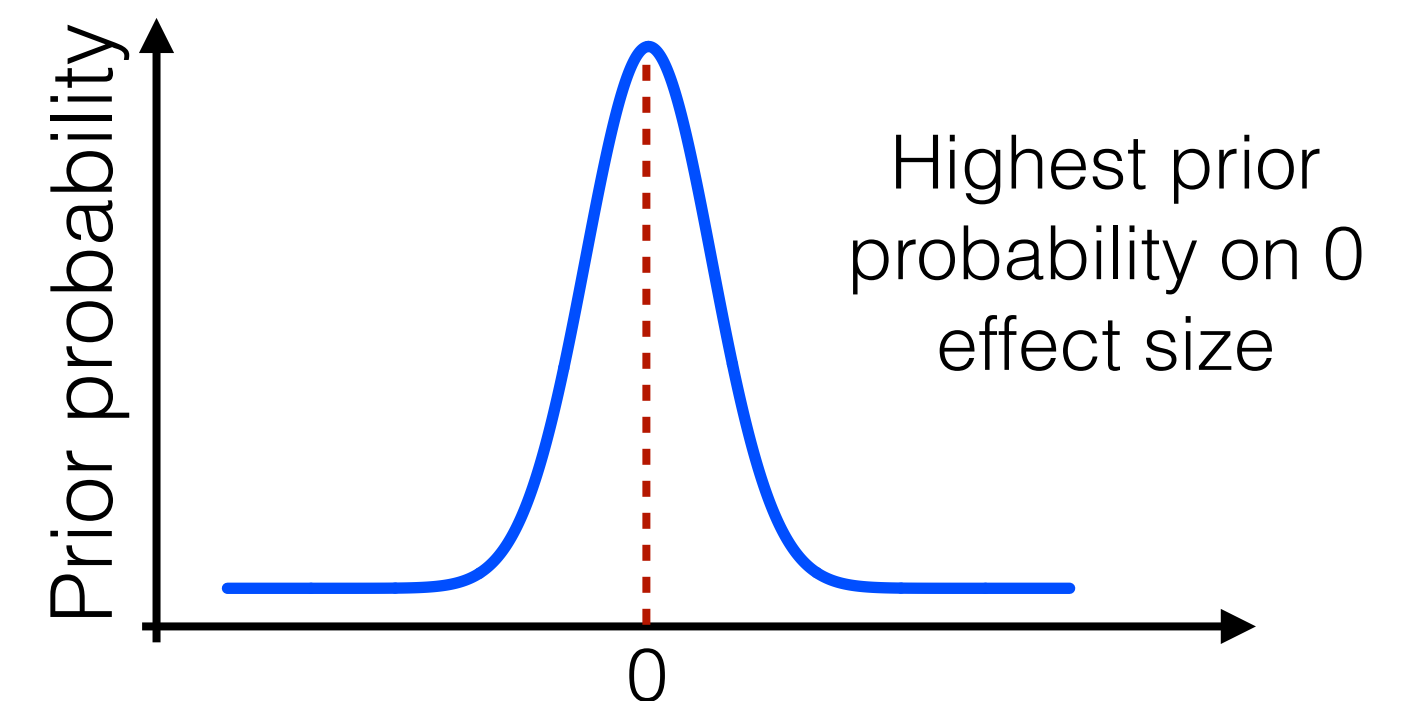
Intercept: $P(b) \sim \mathcal{U}(\text{mouse} | \text{elephant})$



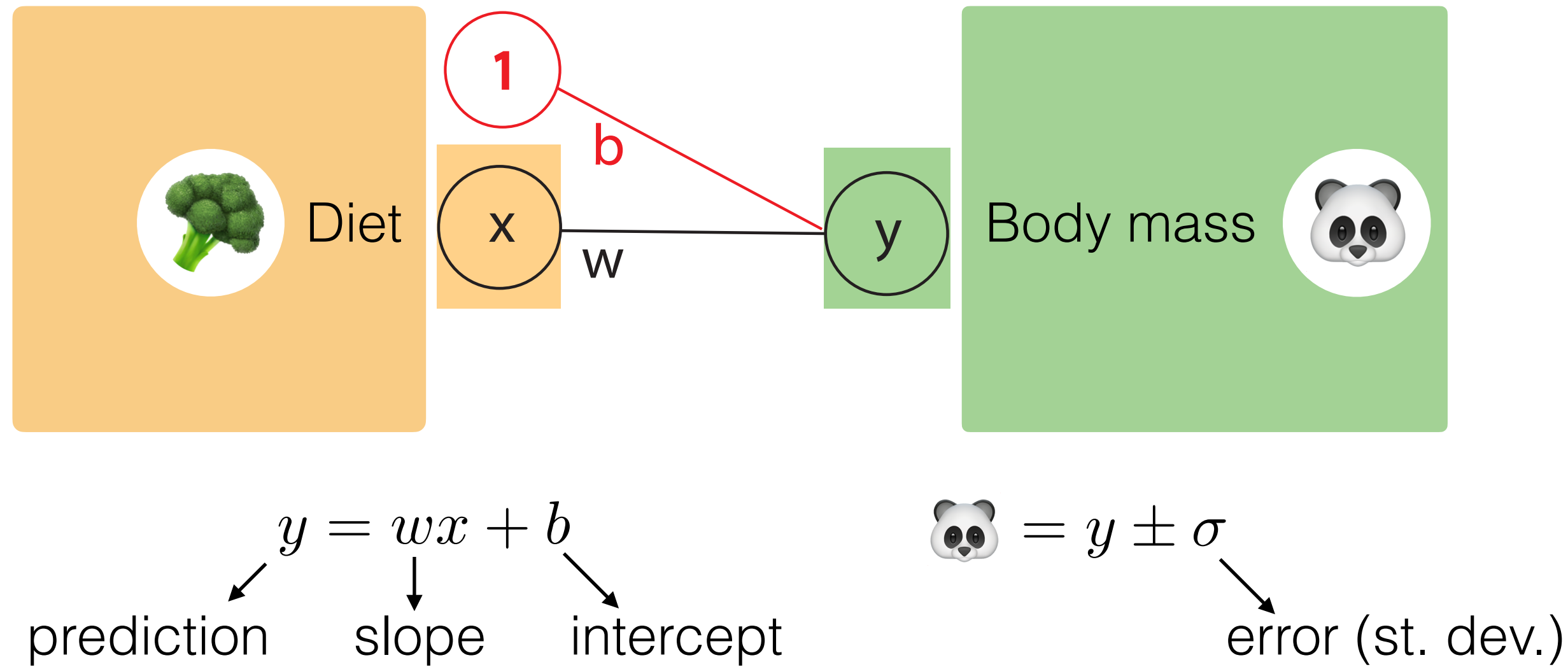
Error: $P(\sigma) \sim \Gamma(\alpha, \beta)$



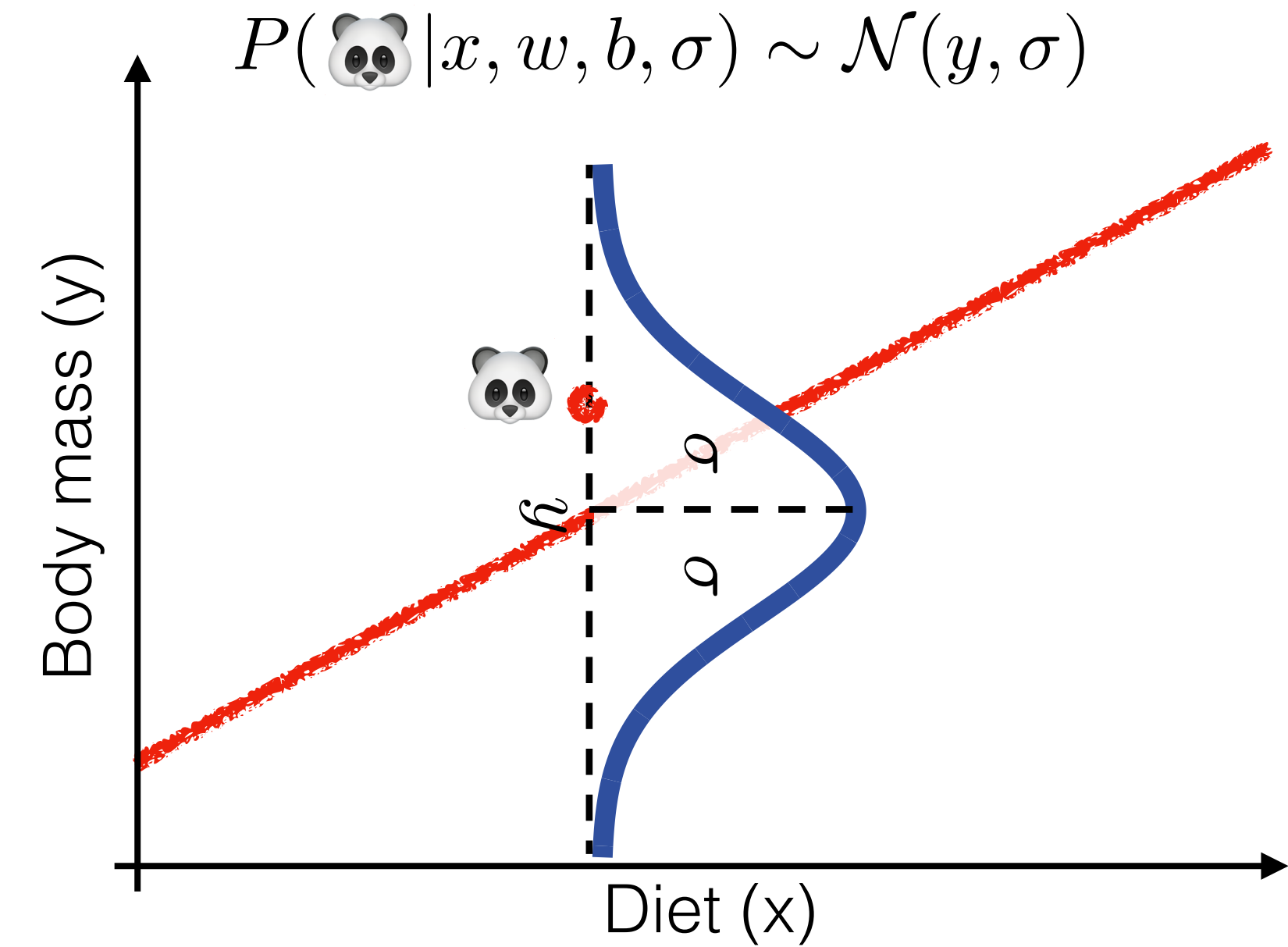
Slope: $P(w) \sim \mathcal{N}(0, 1)$



Linear regression

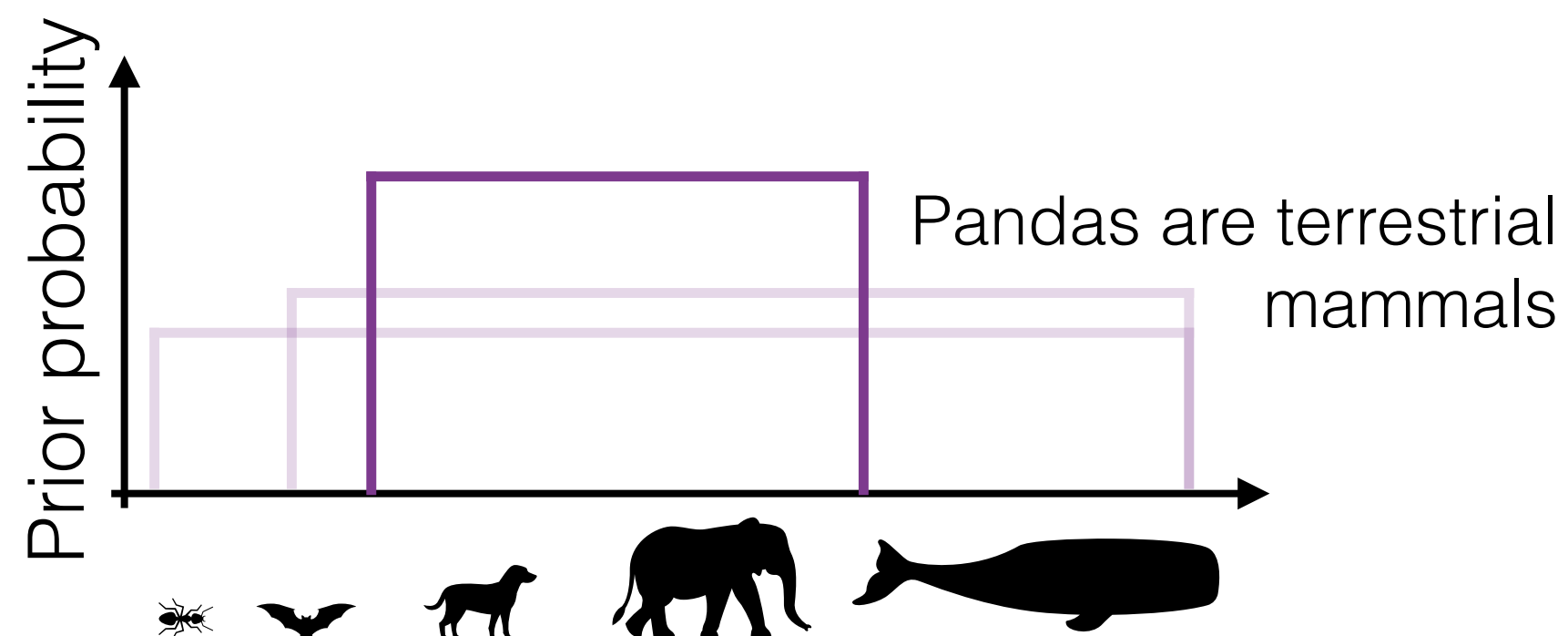


Likelihood of an observed panda body mass

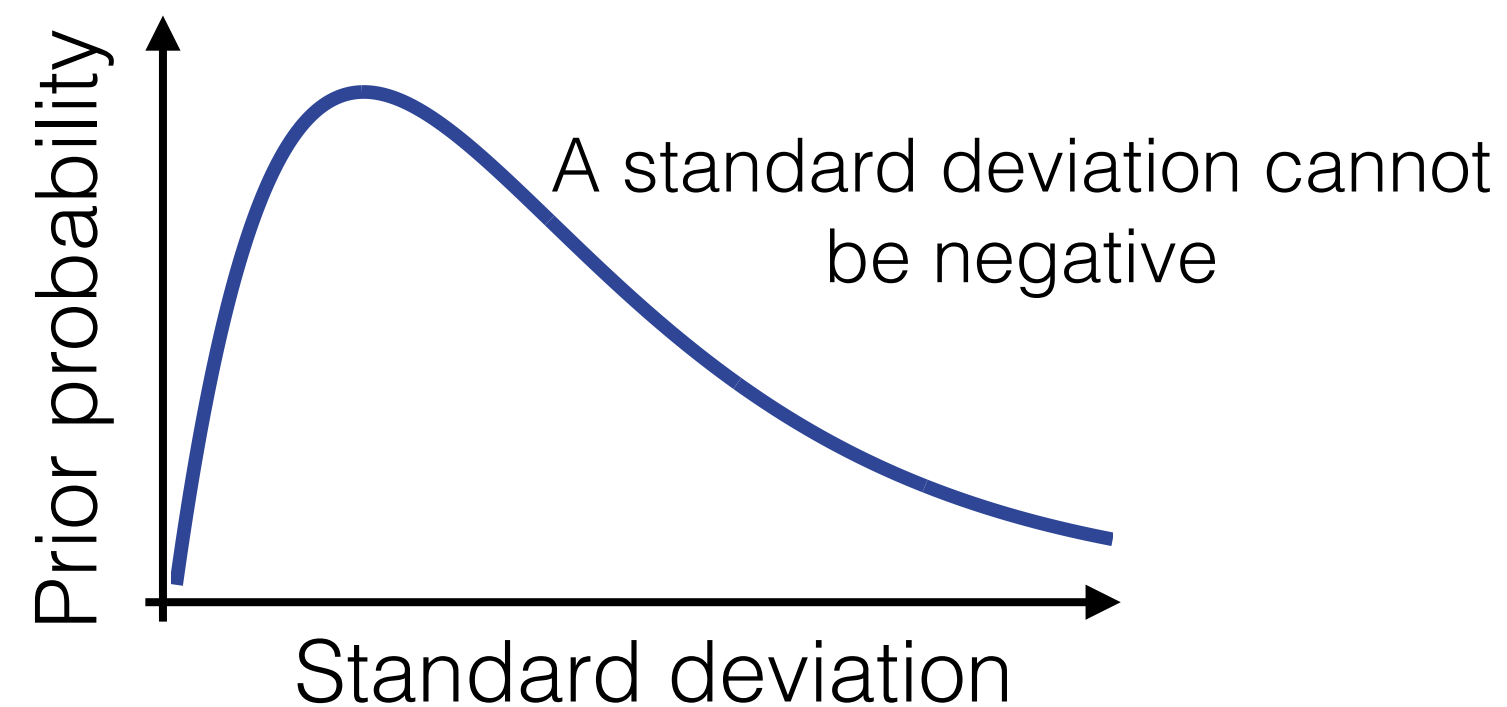


Priors on the model parameters

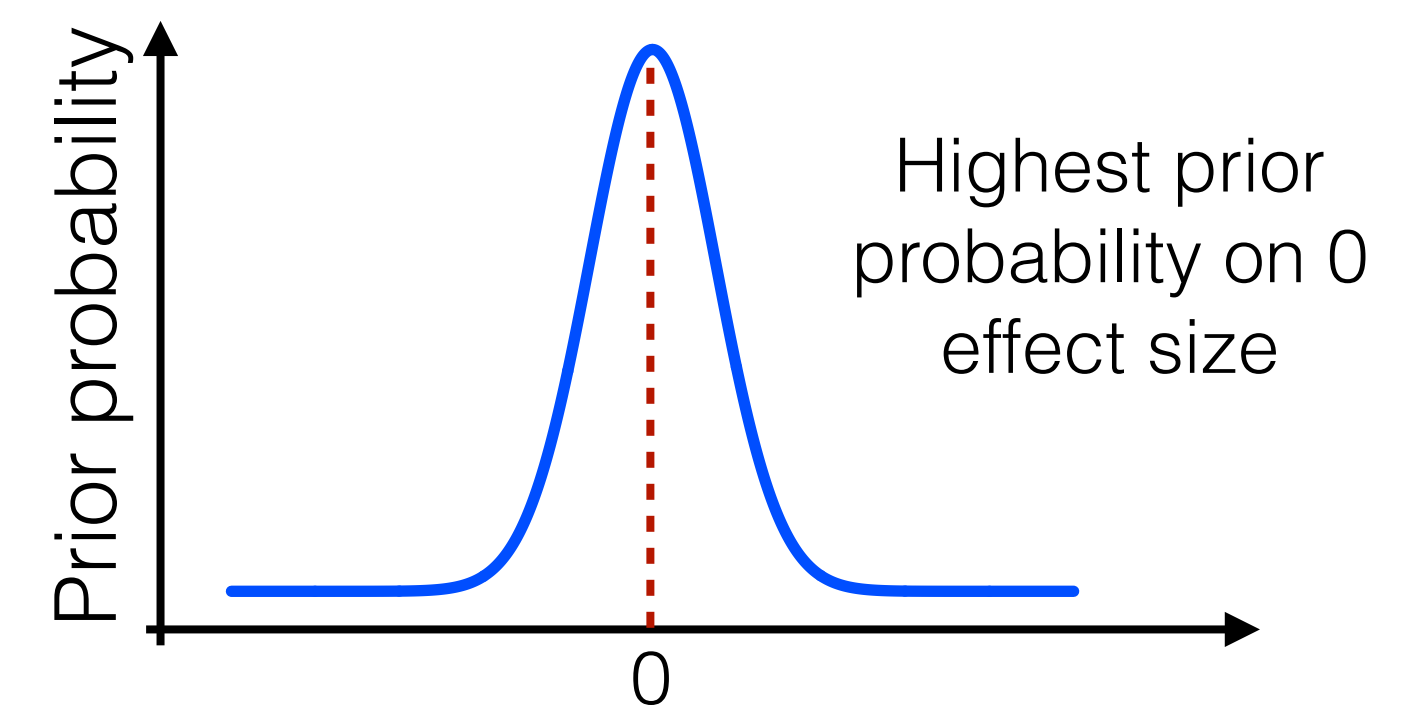
Intercept: $P(b) \sim \mathcal{U}(\text{mouse} | \text{elephant})$



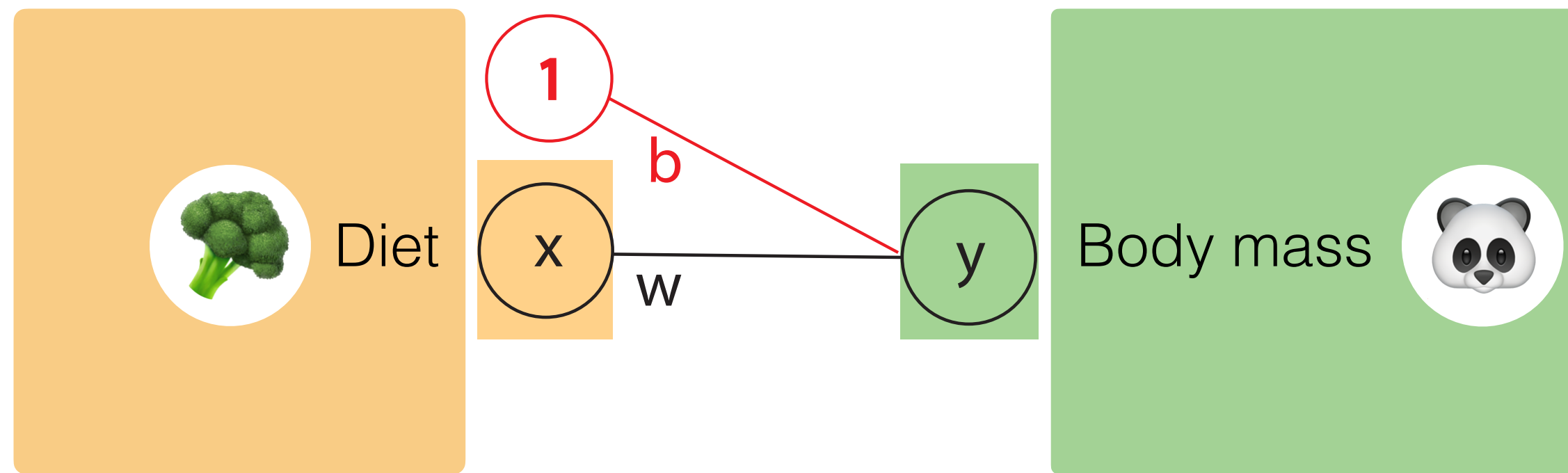
Error: $P(\sigma) \sim \Gamma(\alpha, \beta)$



Slope: $P(w) \sim \mathcal{N}(0, 1)$



Linear regression



$$y = wx + b$$

prediction slope intercept

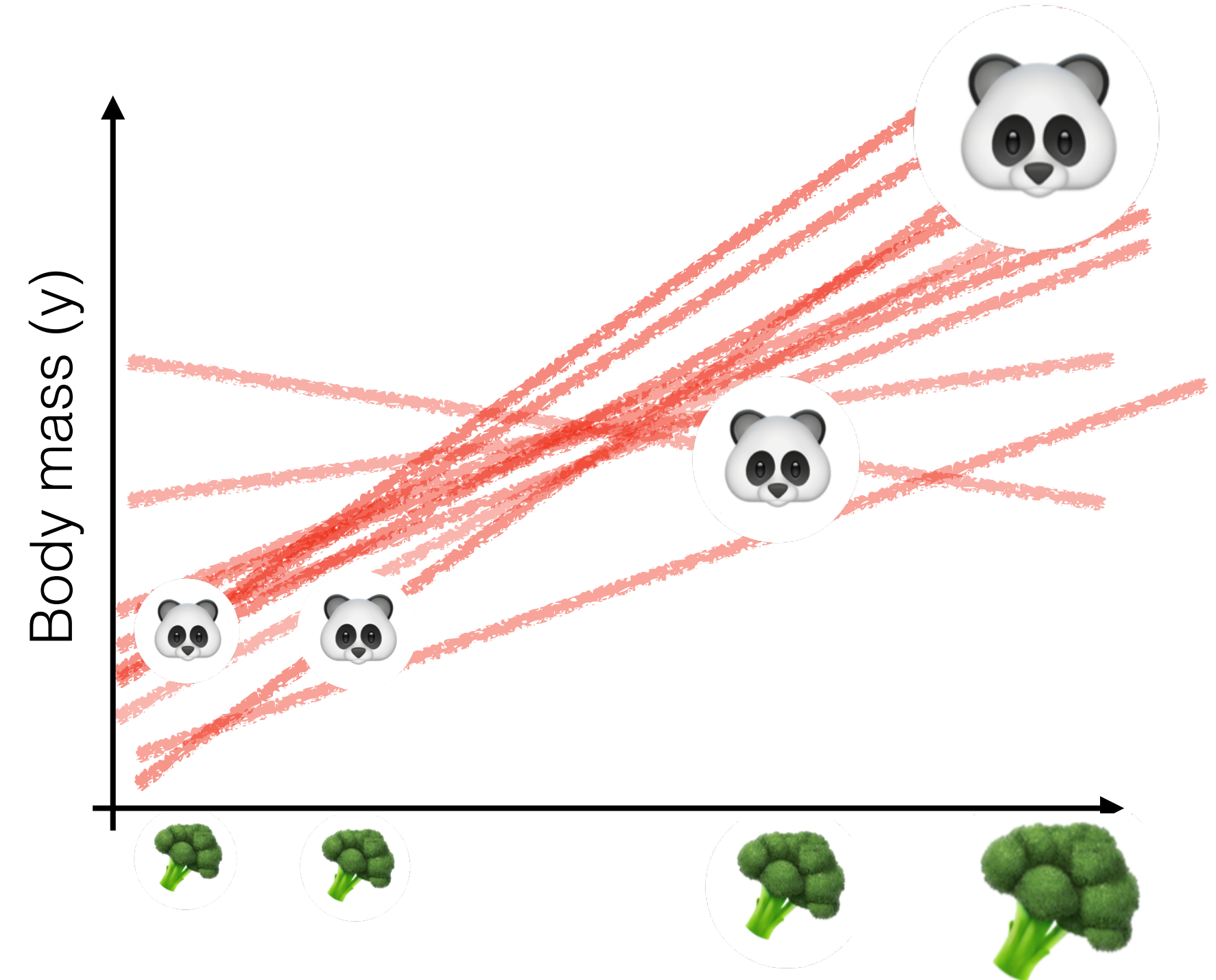
$$\text{panda} = y \pm \sigma$$

error (st. dev.)

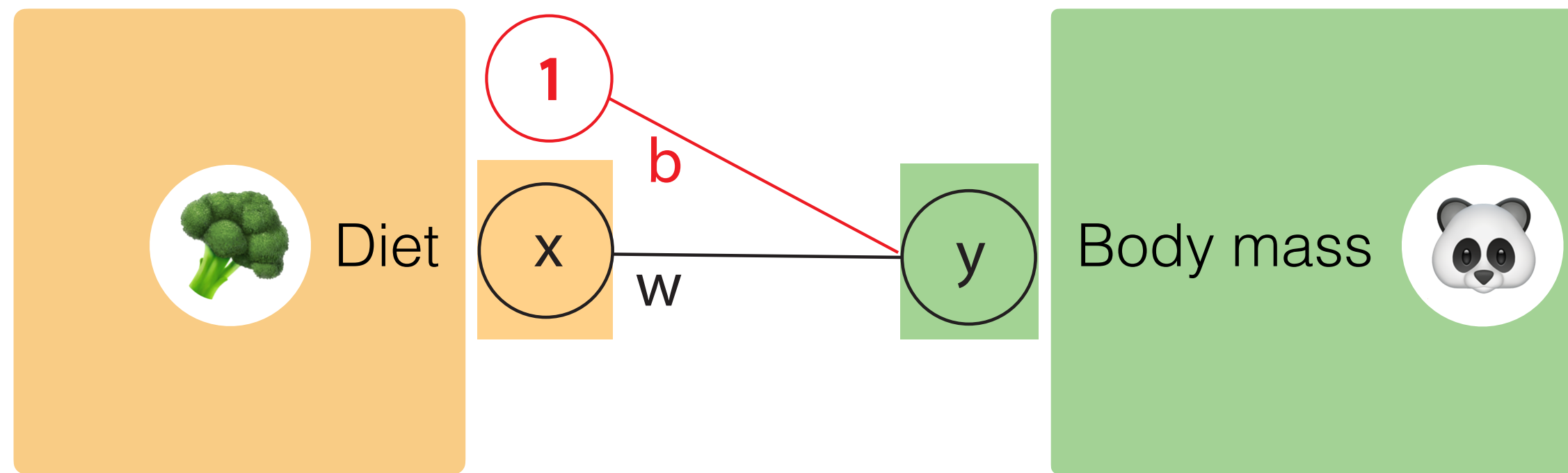
Data ('training set')



Optimizing ('training') a model



Linear regression



$$y = wx + b$$

prediction slope intercept

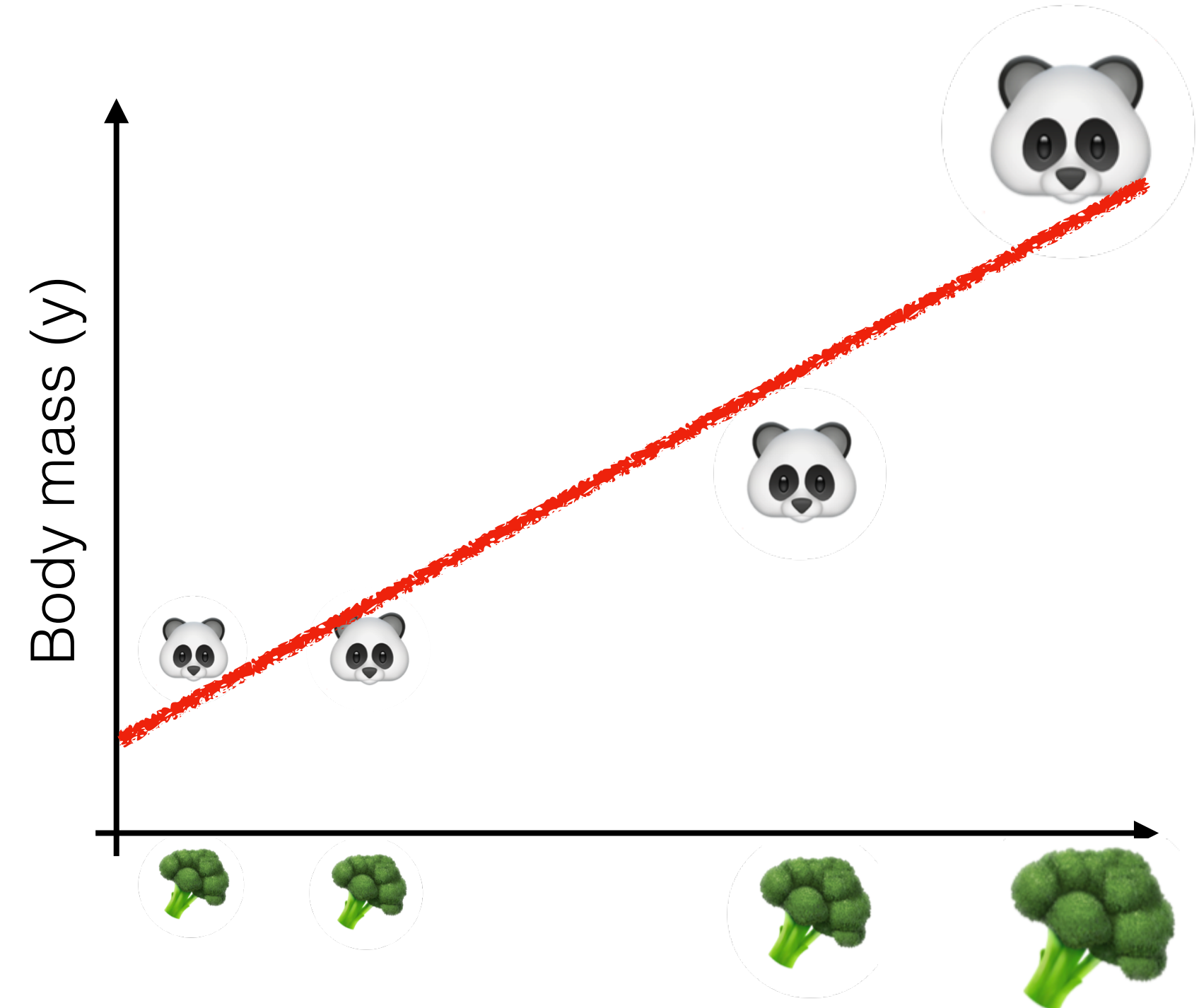
$$\text{panda} = y \pm \sigma$$

error (st. dev.)

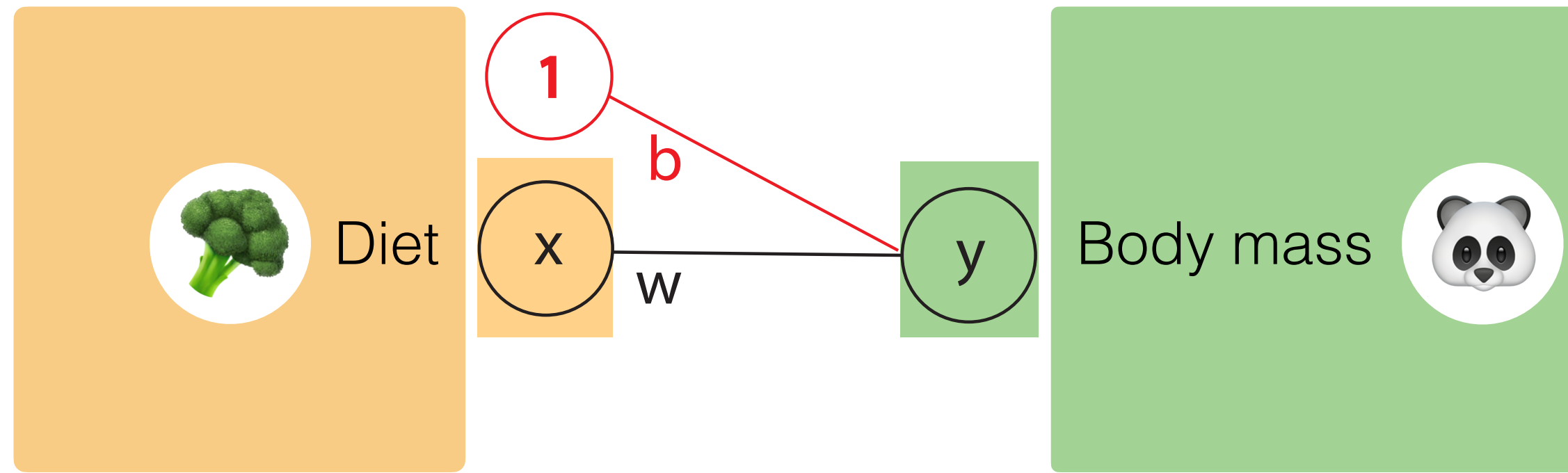
Data ('training set')



Estimated ('trained') model



Linear regression



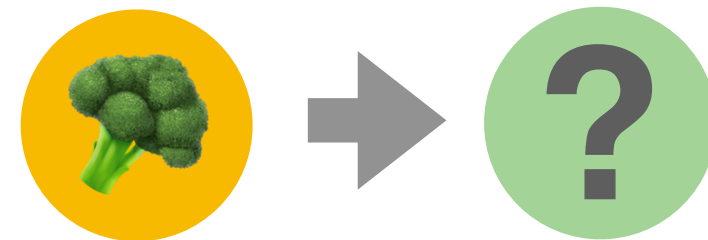
$$y = wx + b$$

prediction slope intercept

$$\text{panda} = y \pm \sigma$$

error (st. dev.)

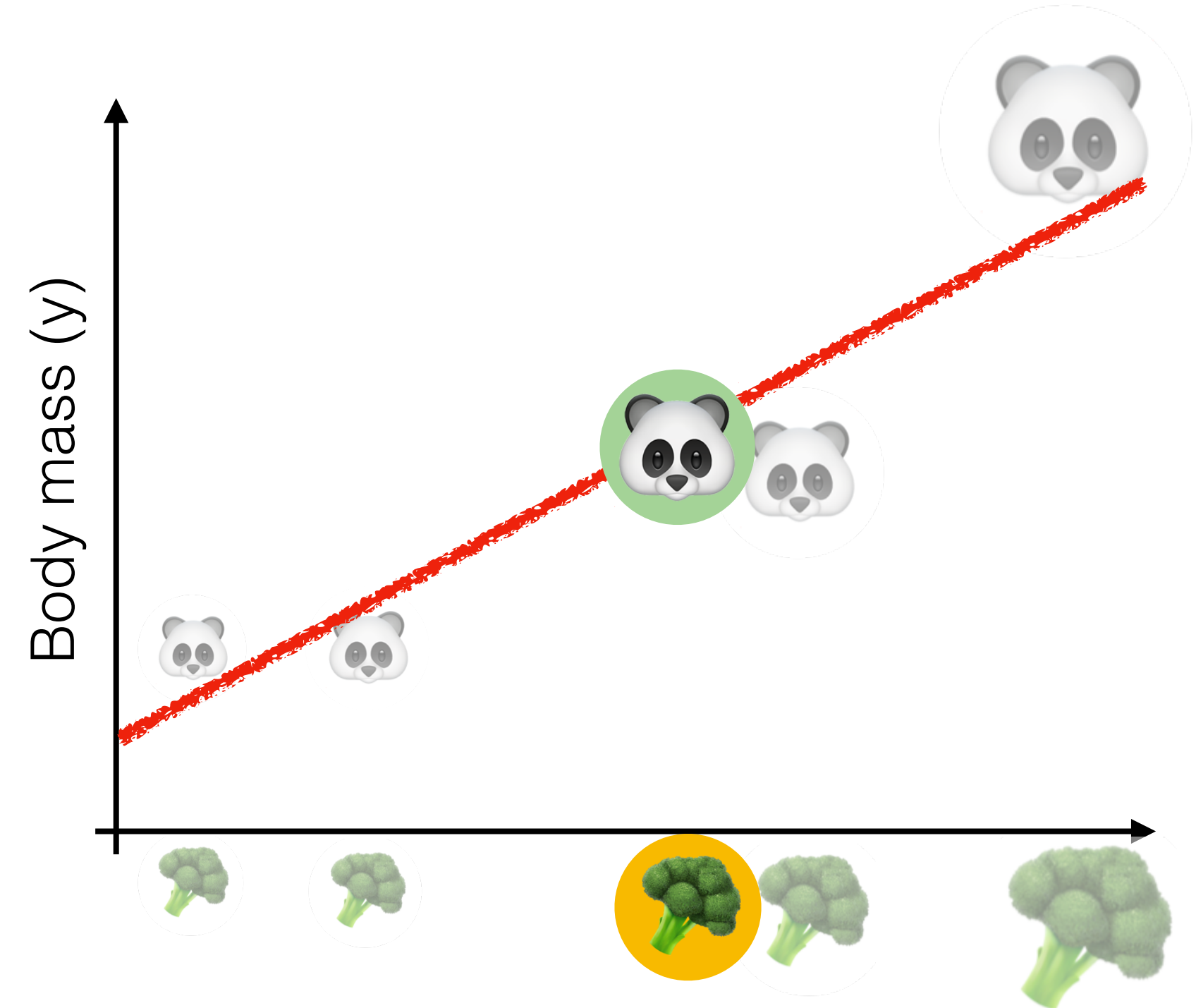
Predicting body mass from diet



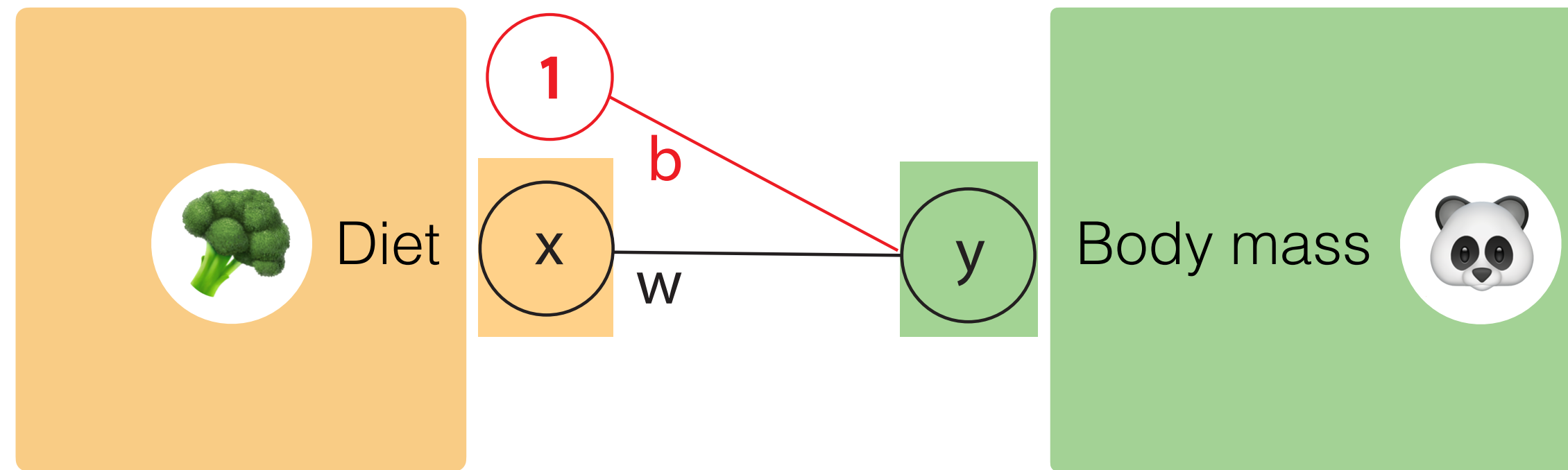
Data ('training set')



Estimated ('trained') model

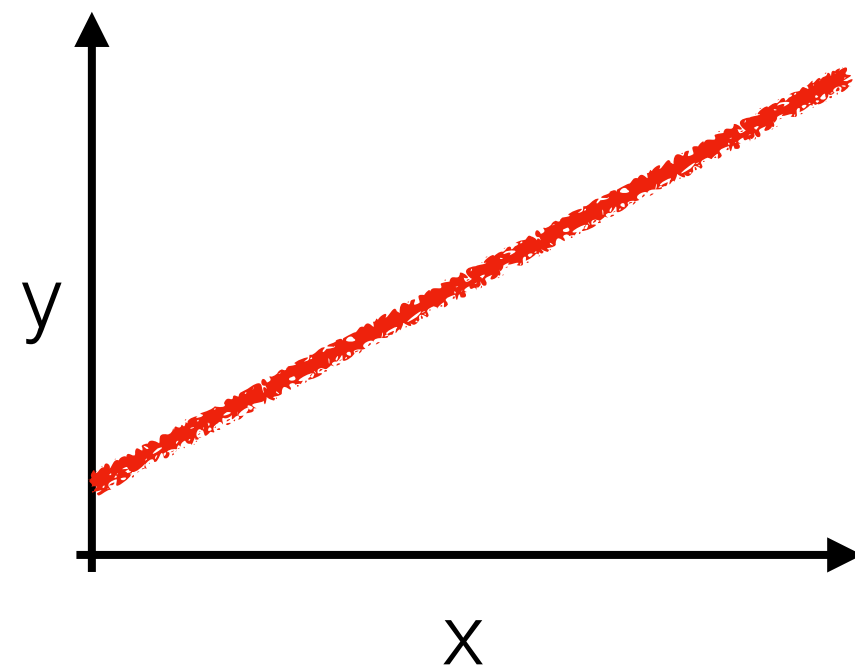


Linear regression

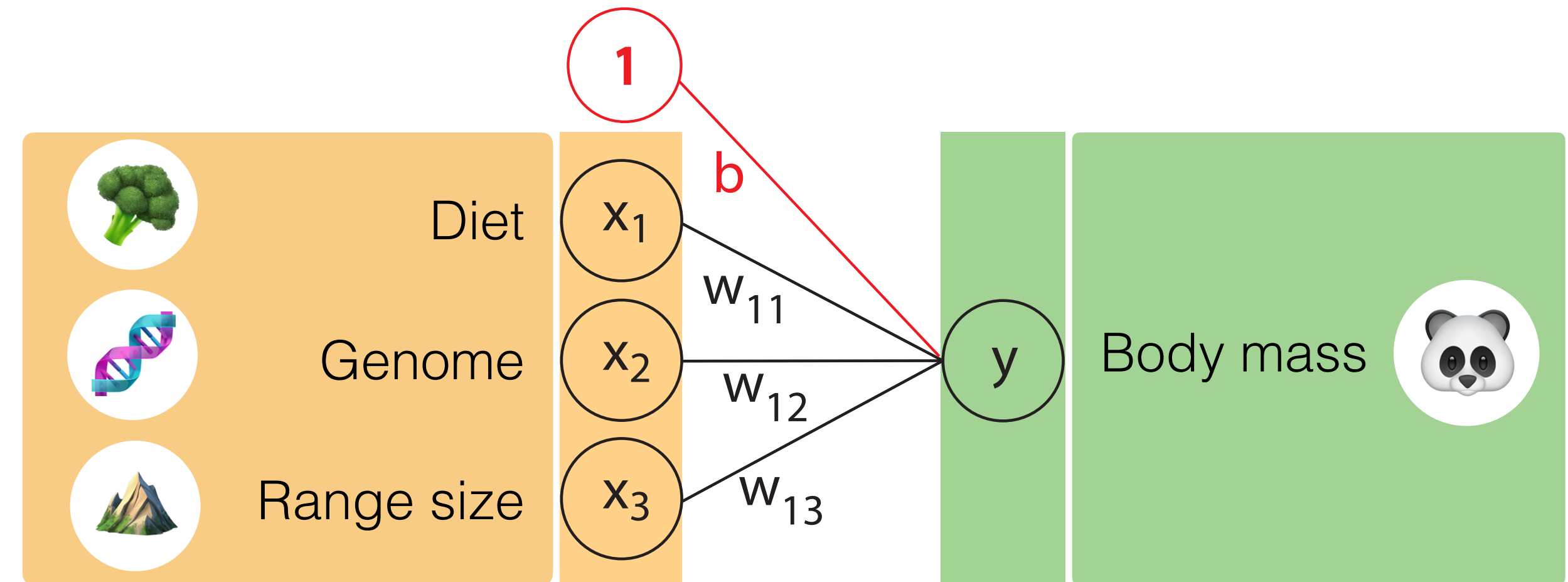


$$y = wx + b$$

prediction slope intercept



Multiple linear regression



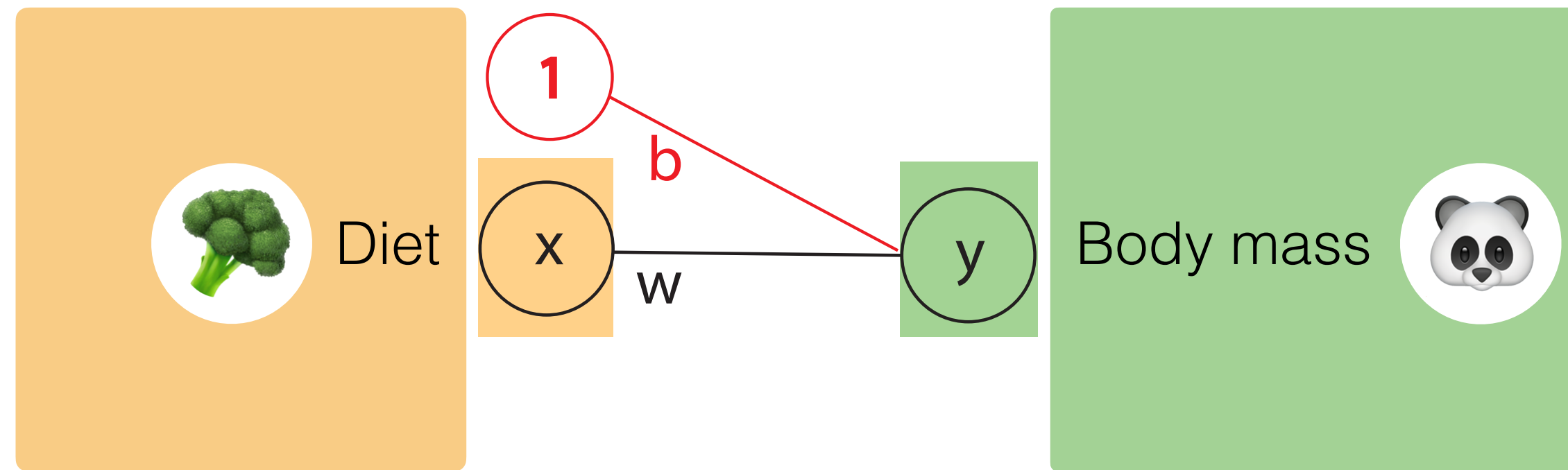
$$y = w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + b$$

prediction slopes (effect sizes) intercept

$$y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (w_{11} \quad w_{12} \quad w_{13}) + b$$

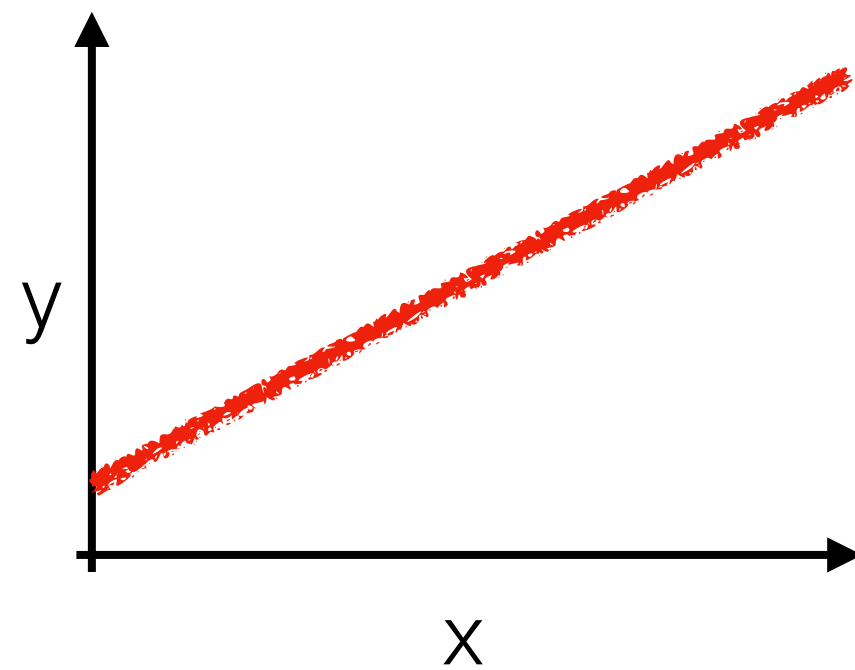
We can re-write it as a matrix multiplication

Linear regression

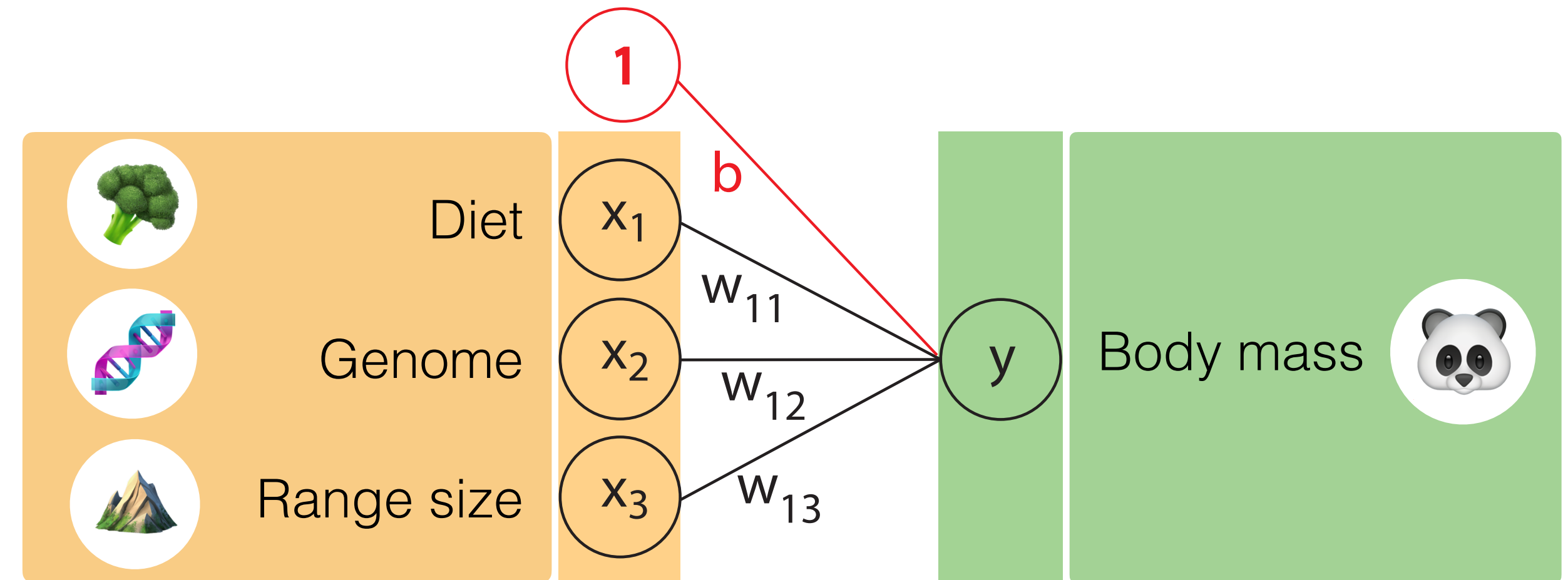


$$y = wx + b$$

prediction slope intercept



Multiple linear regression



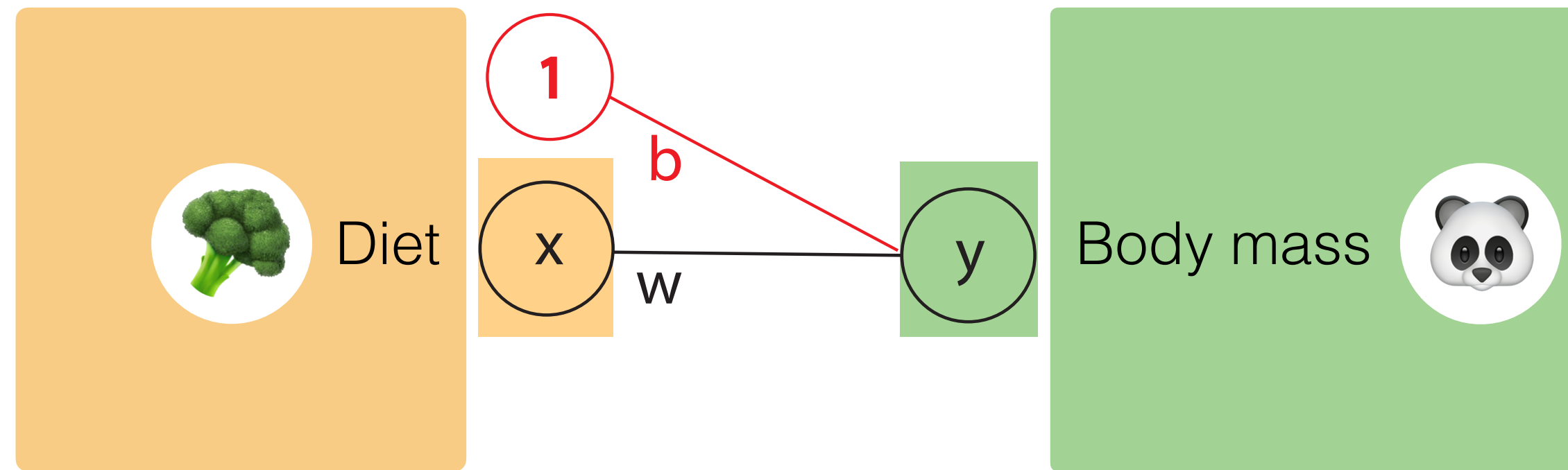
$$y = w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + b$$

prediction slopes (effect sizes) intercept

$$y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & w_{13} \end{pmatrix} + b$$

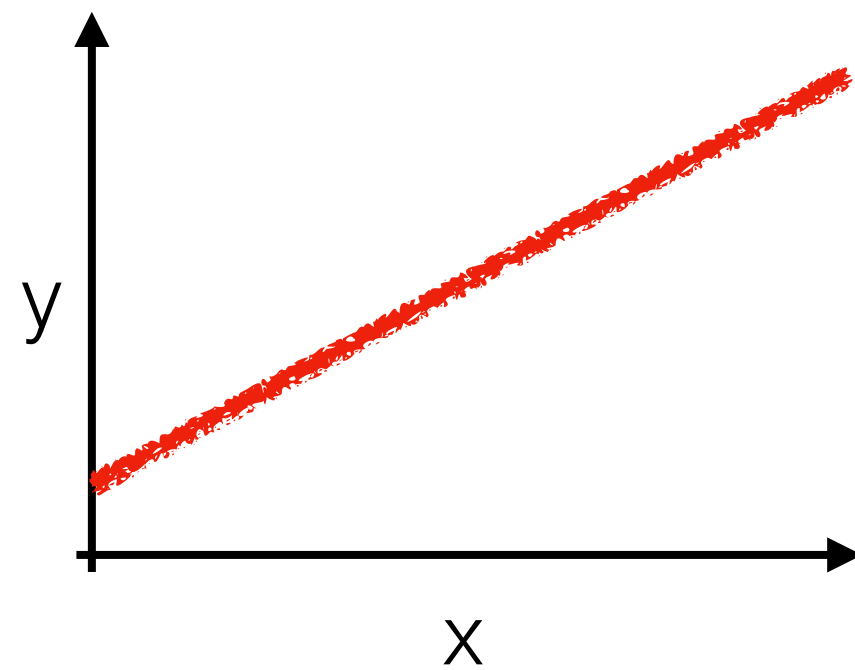
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Linear regression

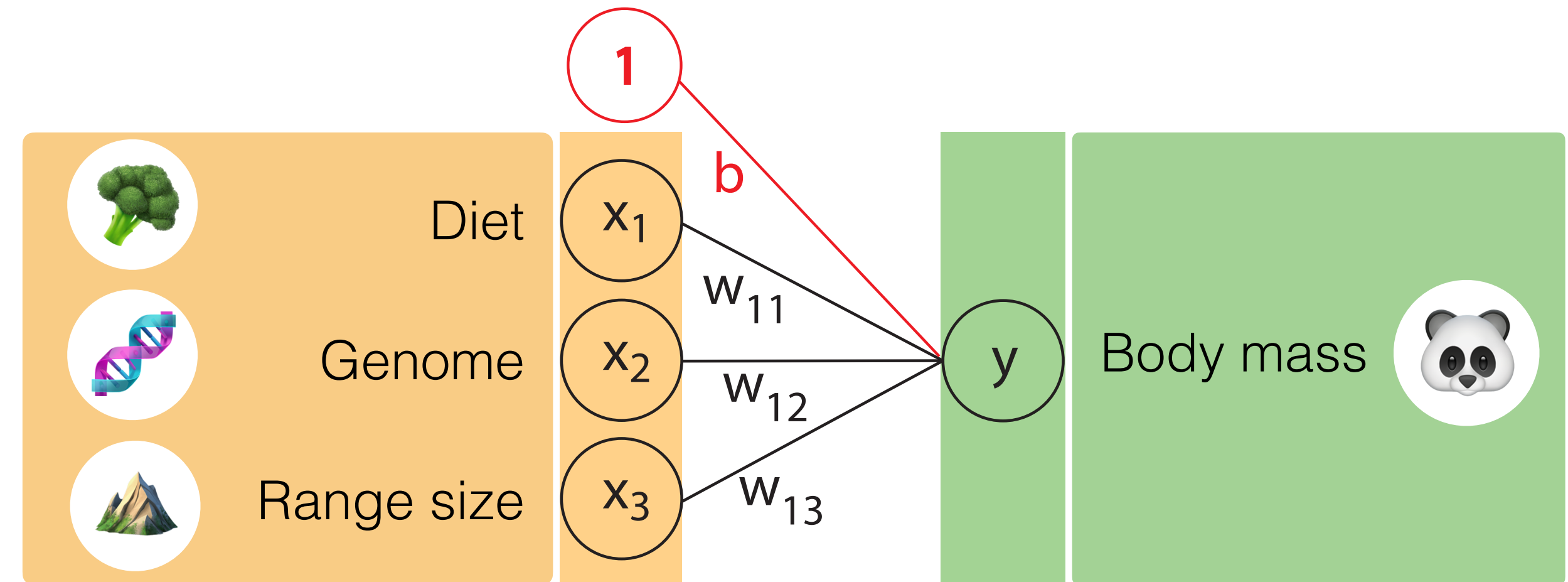


$$y = wx + b$$

prediction slope intercept



Multiple linear regression



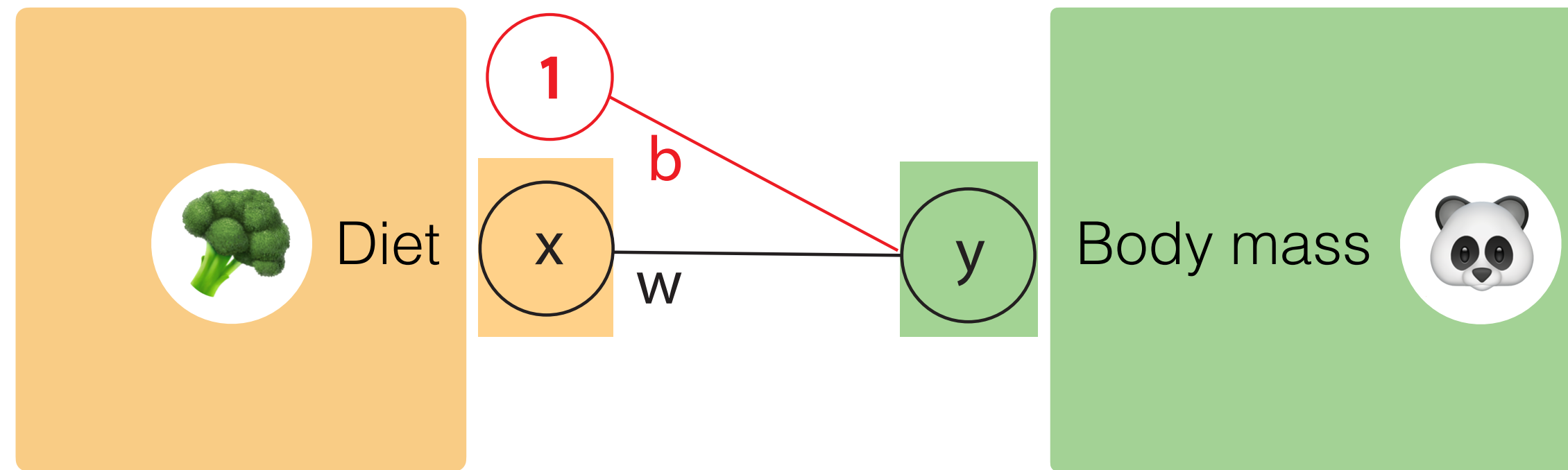
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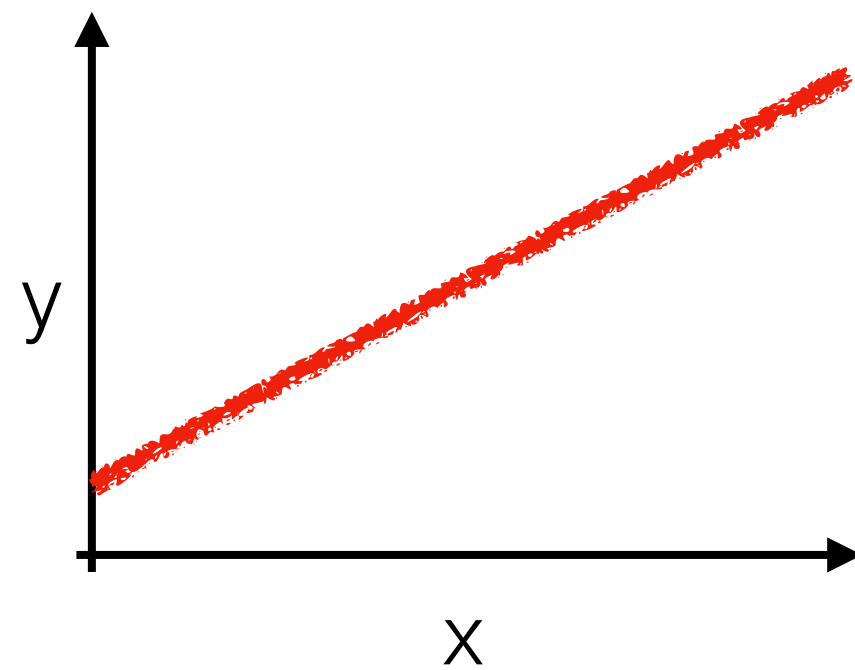
We can re-write it as a matrix multiplication

Linear regression

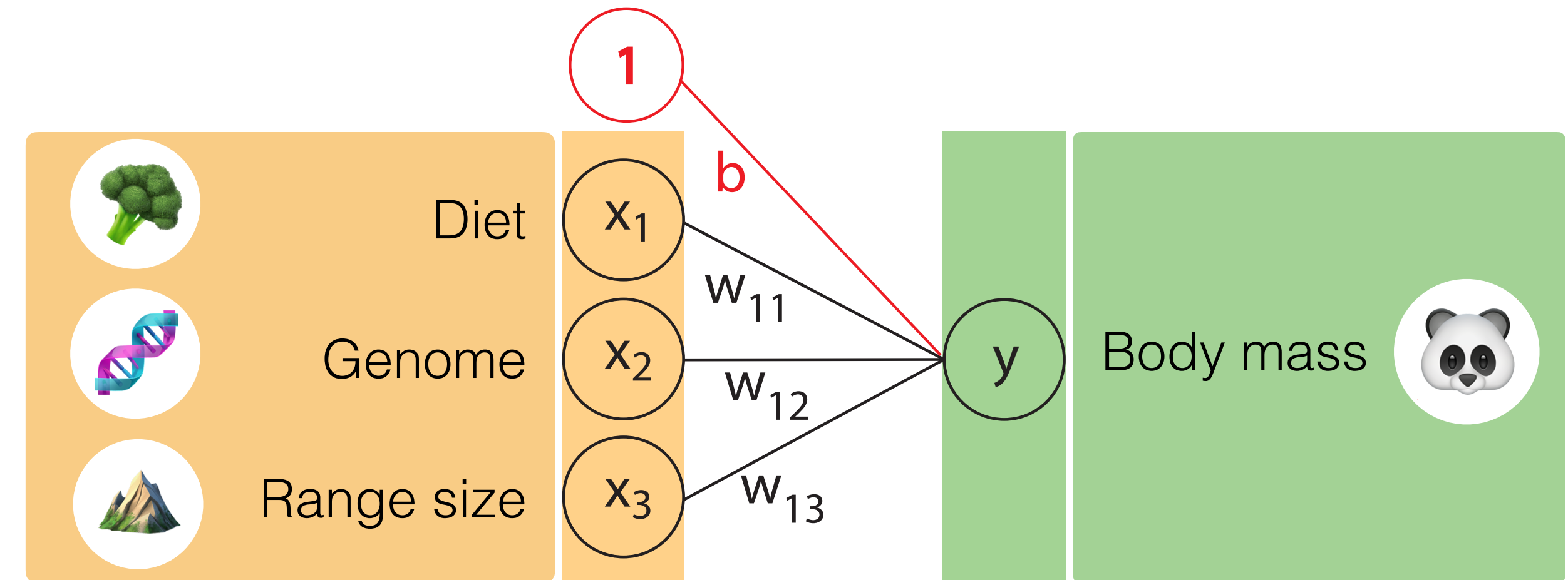


$$y = wx + b$$

prediction slope intercept



Multiple linear regression



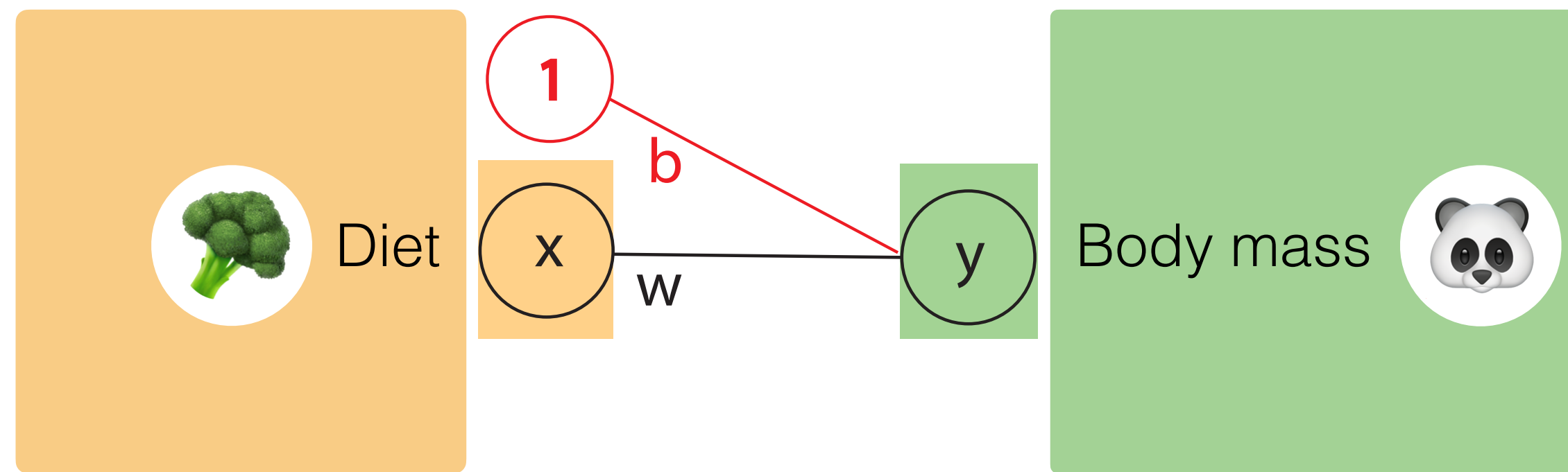
$$y = w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + b$$

prediction slopes (effect sizes) intercept

$$y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (w_{11} \quad w_{12} \quad w_{13}) + b$$

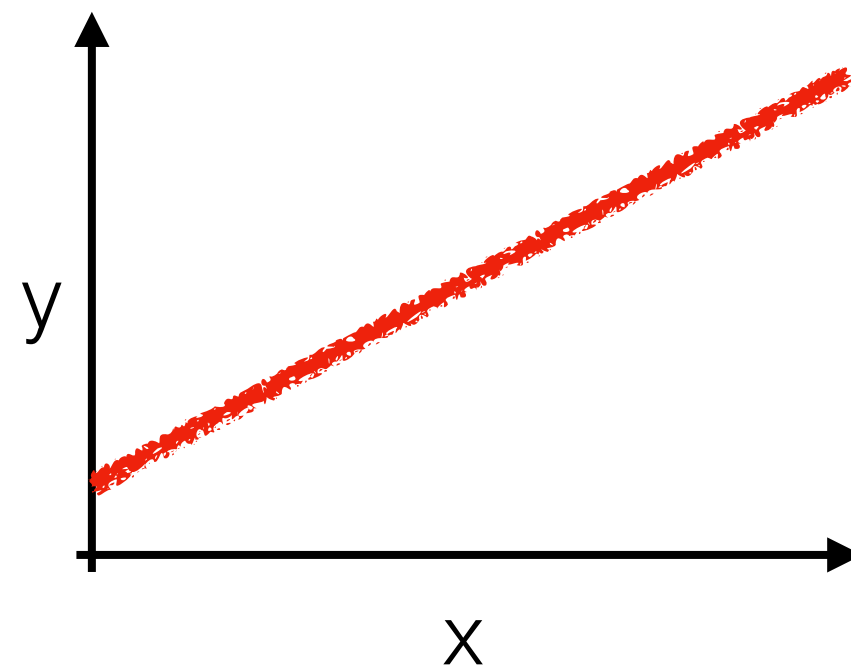
We can re-write it as a matrix multiplication

Linear regression

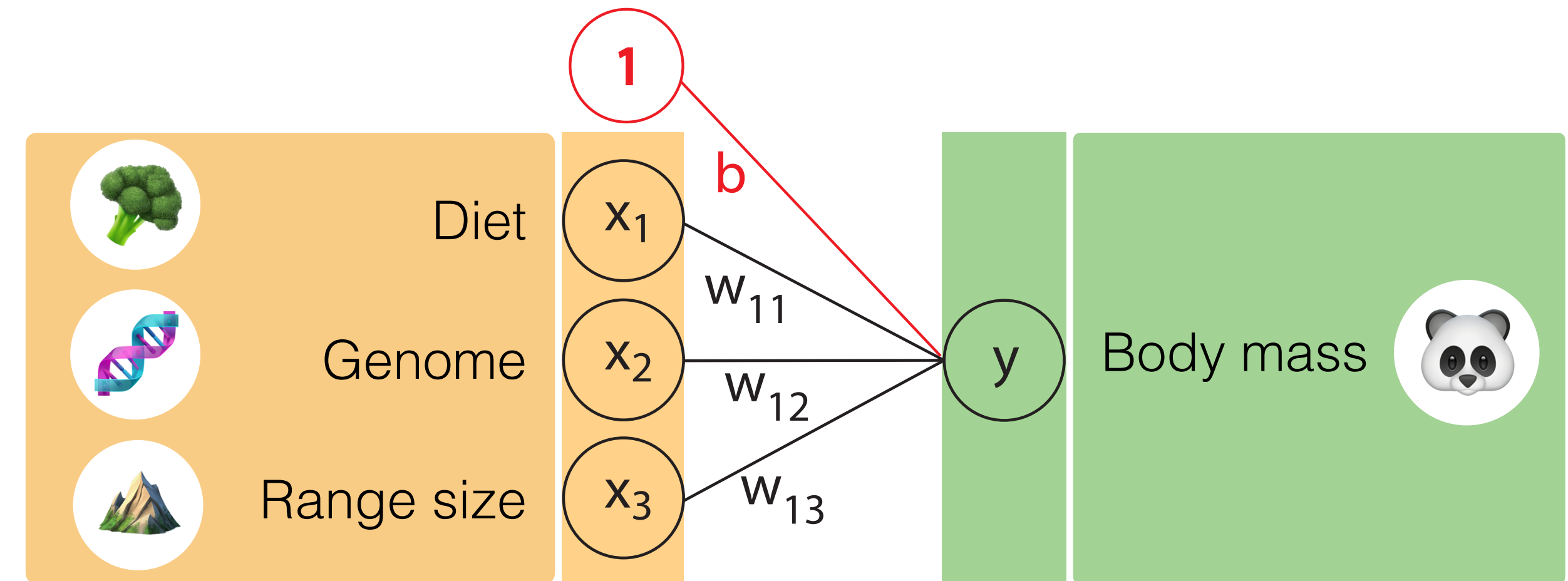


$$y = wx + b$$

prediction slope intercept

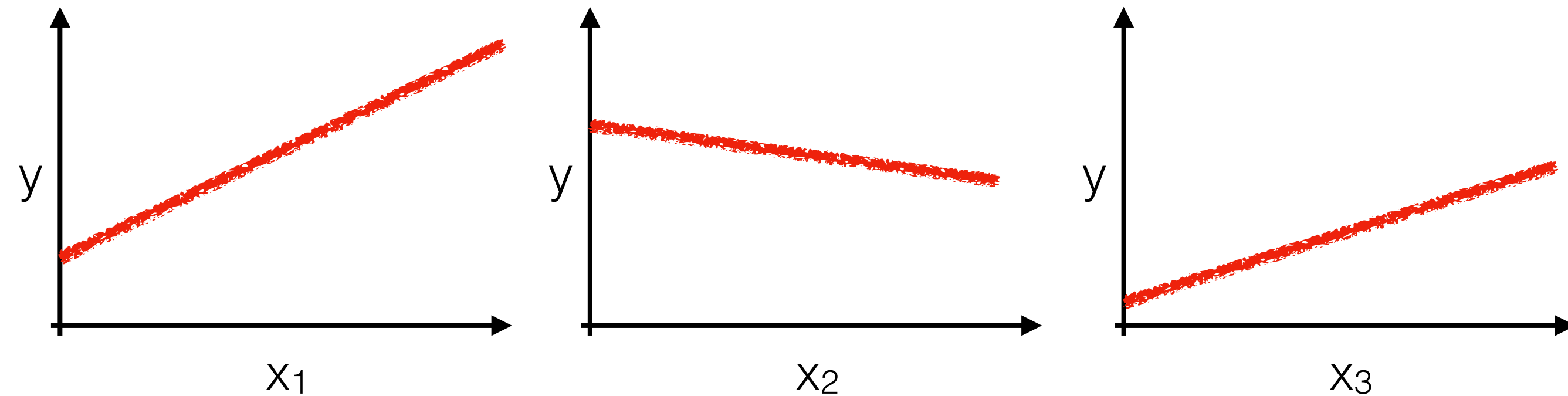


Multiple linear regression

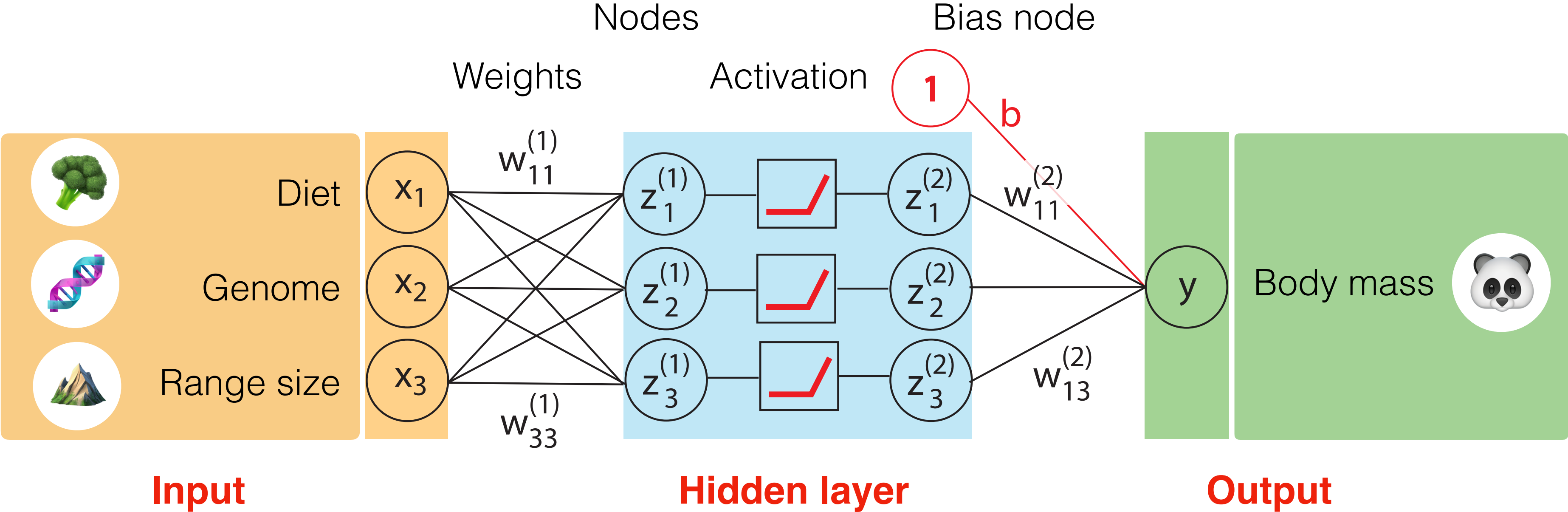


$$y = w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + b$$

prediction slopes (effect sizes) intercept



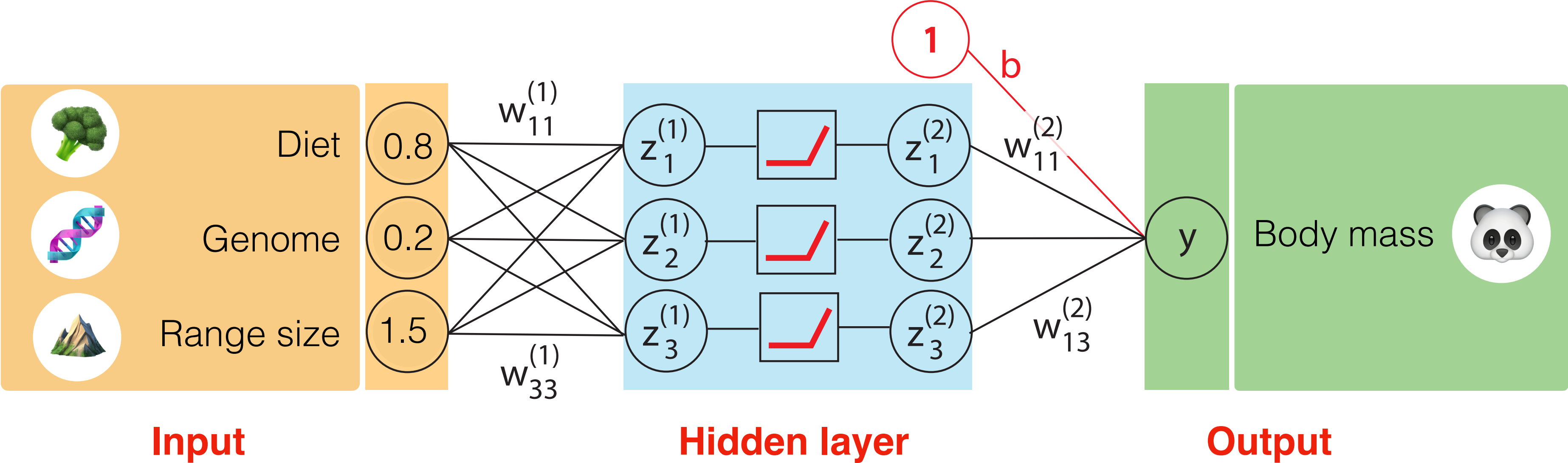
A neural network regression model



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \\ z_3^{(1)} \end{pmatrix}$$

x $w^{(1)}$ $z^{(1)}$

A neural network regression model

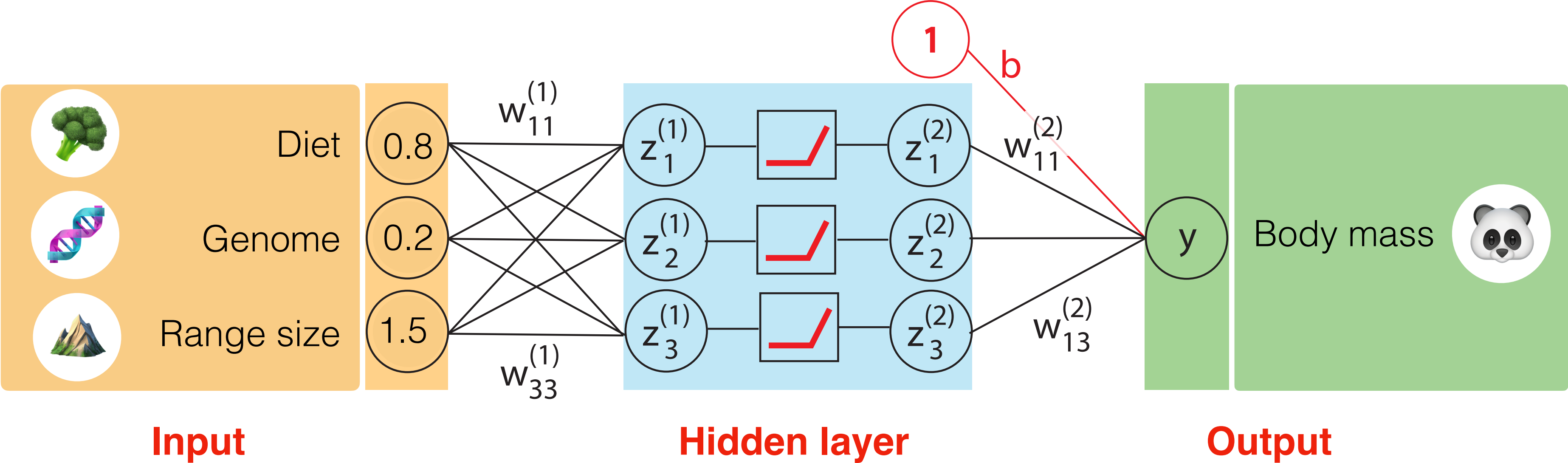


Matrix multiplication
Multiply each x by w columns

$$\begin{pmatrix} 0.8 \\ 0.2 \\ 1.5 \end{pmatrix} \begin{pmatrix} -0.4 & 1.1 & 1.1 \\ 0.9 & 0.4 & -1.2 \\ 0.6 & 1.3 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 & 1.6 \\ 0.7 & 0.1 & -1.8 \\ 0.5 & 0.3 & 0.3 \end{pmatrix}$$

x $w^{(1)}$

A neural network regression model

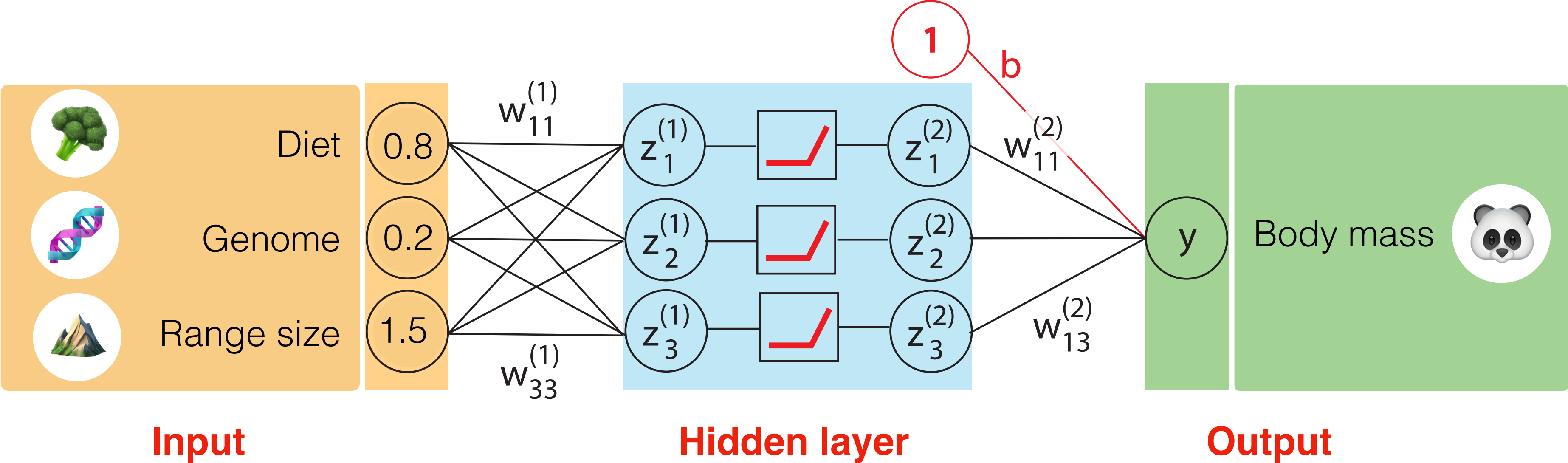


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x $w^{(1)}$

A neural network regression model

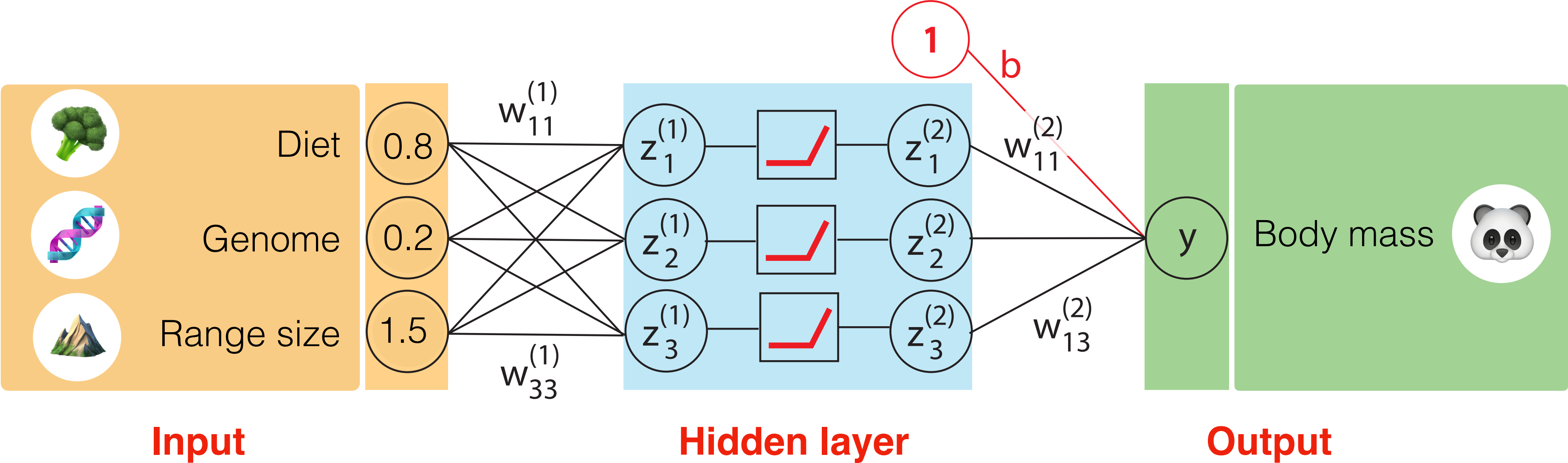


Matrix multiplication
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x $w^{(1)}$

A neural network regression model

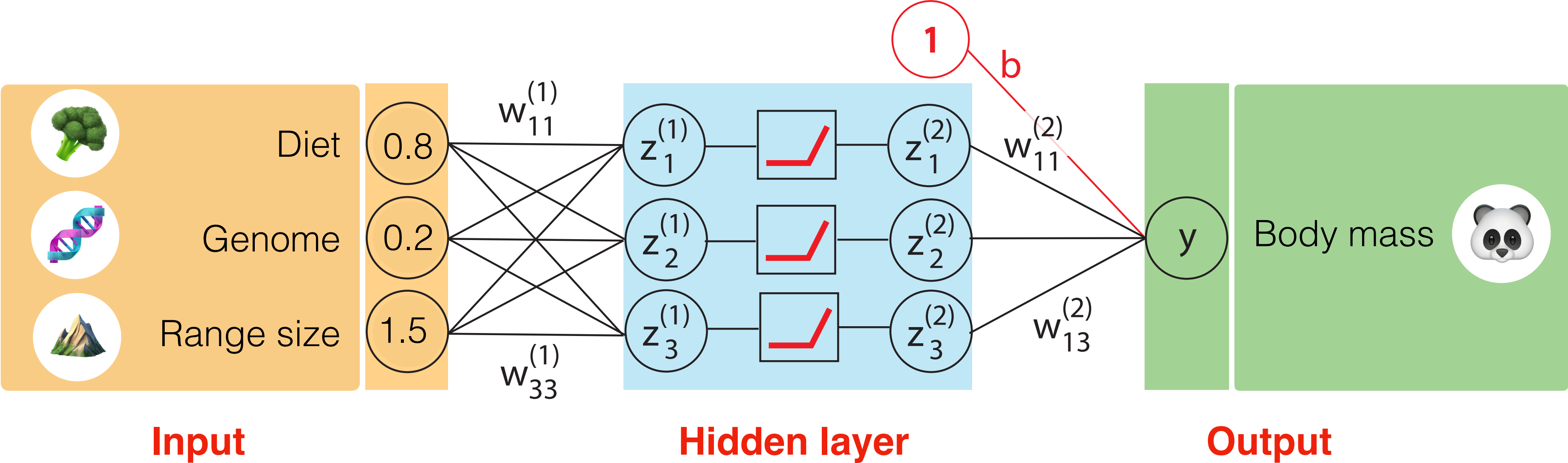


Matrix multiplication
Multiply each x by w columns
and sum by rows.

$$\begin{pmatrix} 0.8 \\ 0.2 \\ 1.5 \end{pmatrix} \begin{pmatrix} -0.4 & 1.1 & 1.1 \\ 0.9 & 0.4 & -1.2 \\ 0.6 & 1.3 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 & 1.6 \\ 0.7 & 0.1 & -1.8 \\ 0.5 & 0.3 & 0.3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.0 \\ 1.1 \end{pmatrix}$$

x $w^{(1)}$ $z^{(1)}$

A neural network regression model

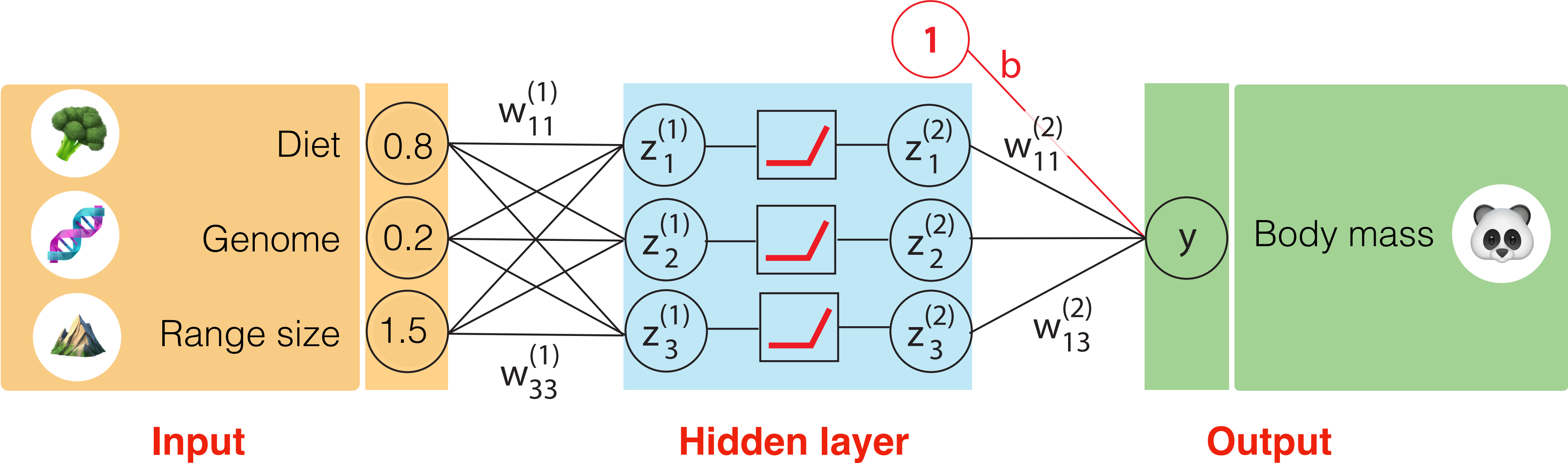


Matrix multiplication
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x $w^{(1)}$ $z^{(1)}$

A neural network regression model

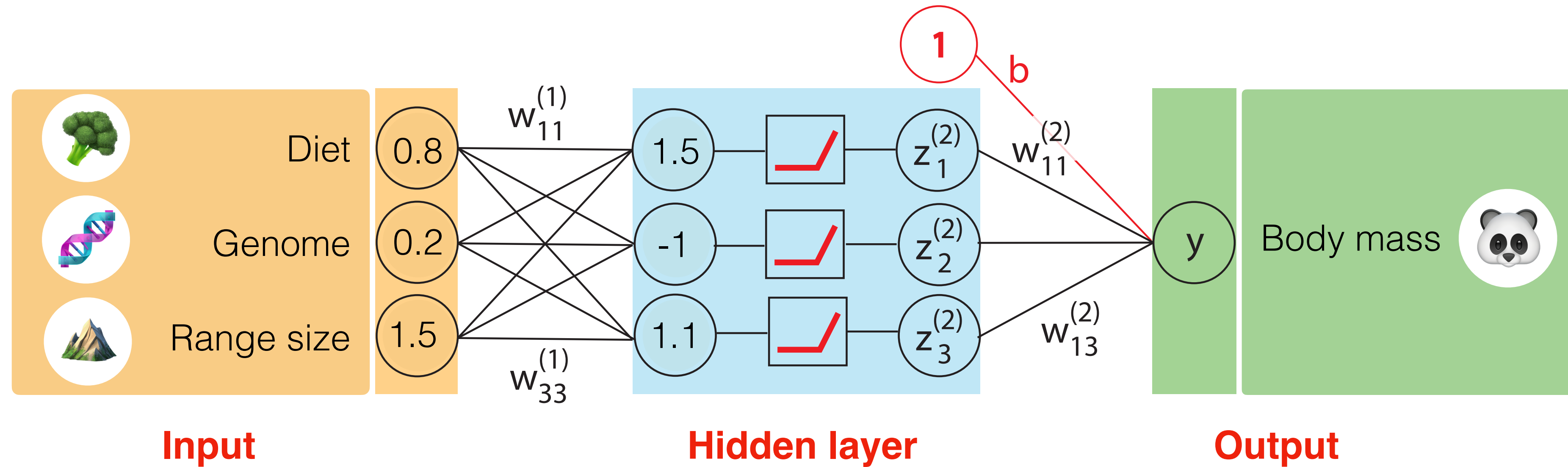


Matrix multiplication
Multiply each x by w columns
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$$\begin{pmatrix} 0.8 \\ 0.2 \\ 1.5 \end{pmatrix} \begin{pmatrix} -0.4 & 1.1 & 1.1 \\ 0.9 & 0.4 & -1.2 \\ 0.6 & 1.3 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 & 1.6 \\ 0.7 & 0.1 & -1.8 \\ 0.5 & 0.3 & 0.3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.0 \\ 1.1 \end{pmatrix}$$

x $w^{(1)}$ $z^{(1)}$

A neural network regression model



Matrix multiplication

Multiply each x by w columns and sum by rows.

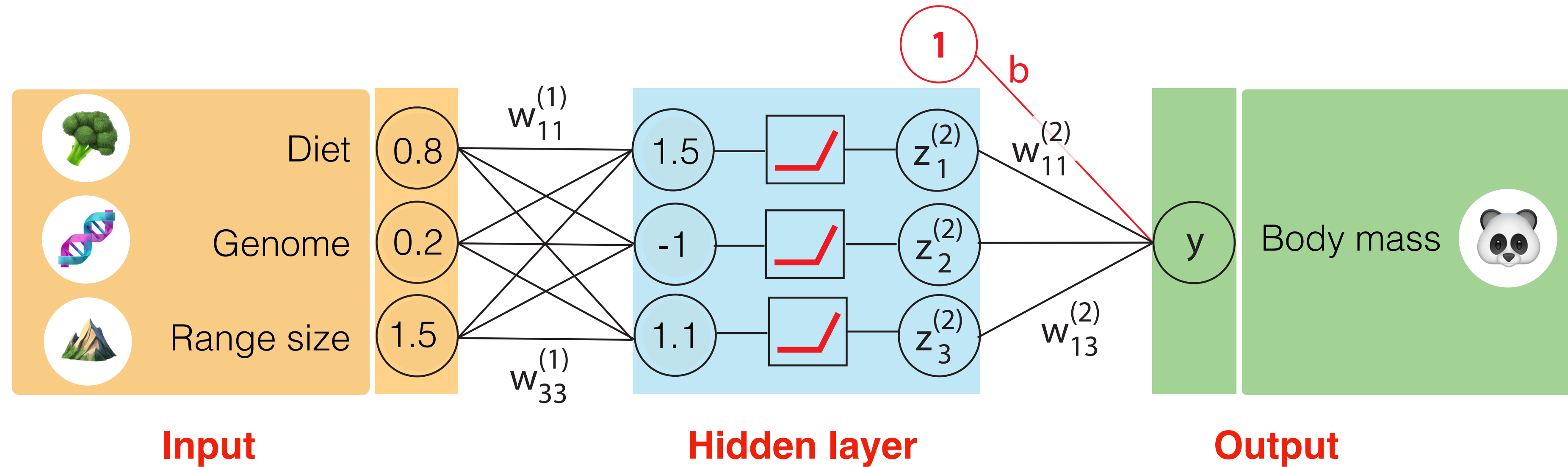
$$\begin{pmatrix} 0.8 \\ 0.2 \\ 1.5 \end{pmatrix} \begin{pmatrix} -0.4 & 1.1 & 1.1 \\ 0.9 & 0.4 & -1.2 \\ 0.6 & 1.3 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 & 1.6 \\ 0.7 & 0.1 & -1.8 \\ 0.5 & 0.3 & 0.3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.0 \\ 1.1 \end{pmatrix}$$

x
 $w^{(1)}$
 $z^{(1)}$

This is comfortably computed in 1 line in Python:

```
z1 = np.einsum('j,ij->i', x, w1)
```

A neural network regression model

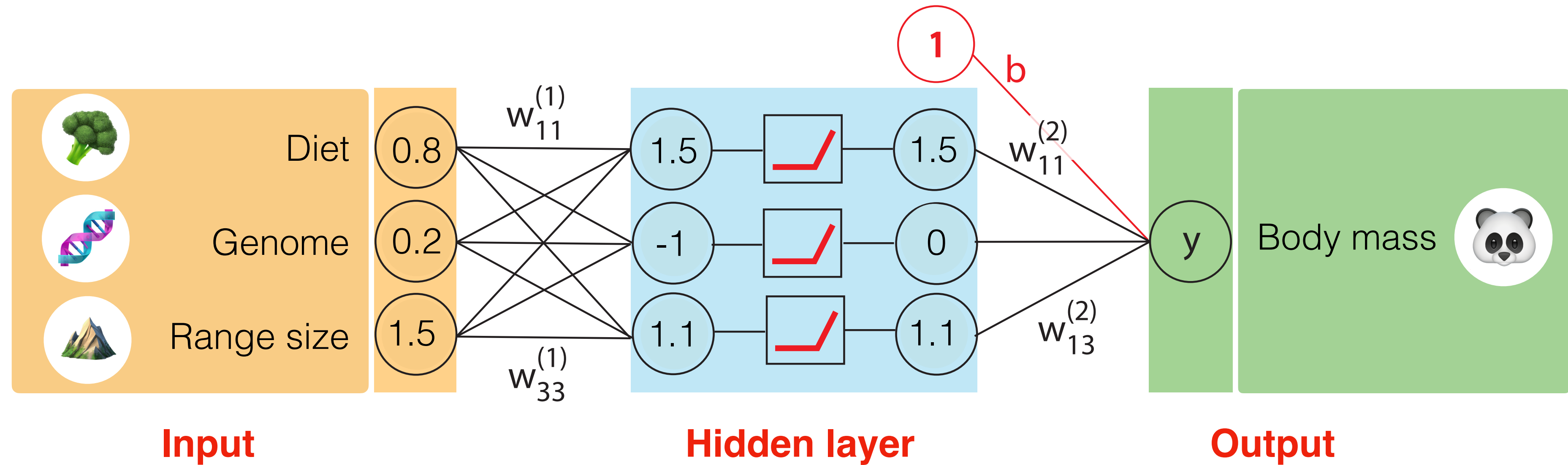


$$\begin{pmatrix} 0.8 \\ 0.2 \\ 1.5 \end{pmatrix} \begin{pmatrix} -0.4 & 1.1 & 1.1 \\ 0.9 & 0.4 & -1.2 \\ 0.6 & 1.3 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 & 1.6 \\ 0.7 & 0.1 & -1.8 \\ 0.5 & 0.3 & 0.3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.0 \\ 1.1 \end{pmatrix}$$

x
 $w^{(1)}$
 $z^{(1)}$

Each item in vector $z^{(1)}$ is now a function of all input features

A neural network regression model



$$\begin{pmatrix} 1.5 \\ -1.0 \\ 1.1 \end{pmatrix} \begin{matrix} \text{ReLU} \end{matrix} = \begin{pmatrix} 1.5 \\ 0 \\ 1.1 \end{pmatrix}$$

$z^{(1)}$ $z^{(2)}$

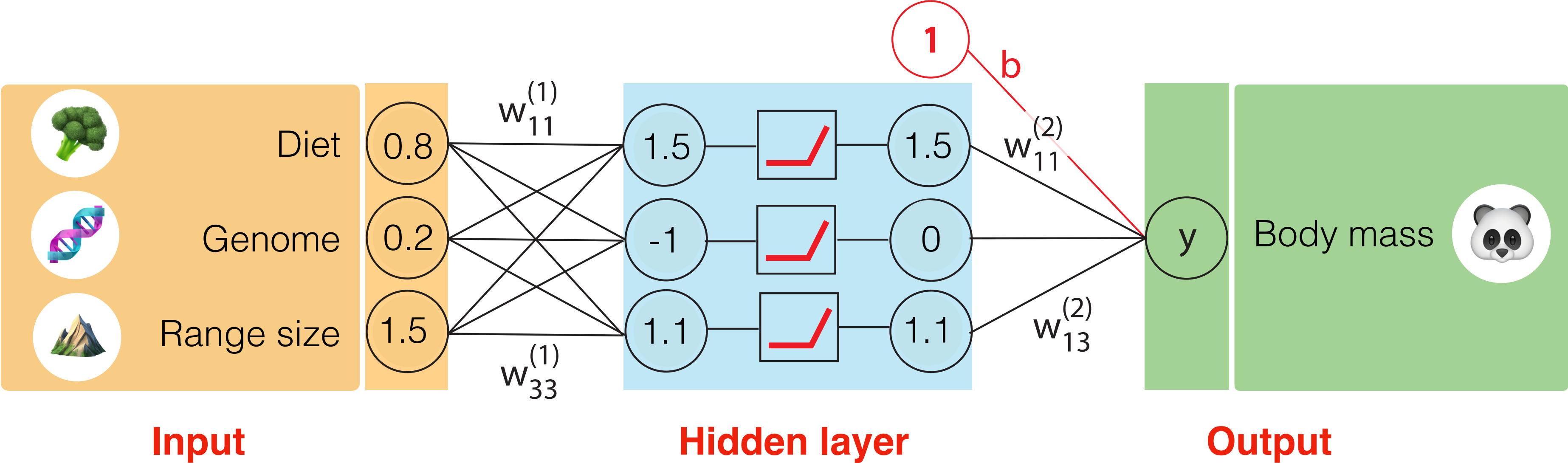
ReLU function

non-linear activation function:
negative values are (arbitrarily)
set to 0

In Python:

$$z2 = z1 * (z1 > 0)$$

A neural network regression model

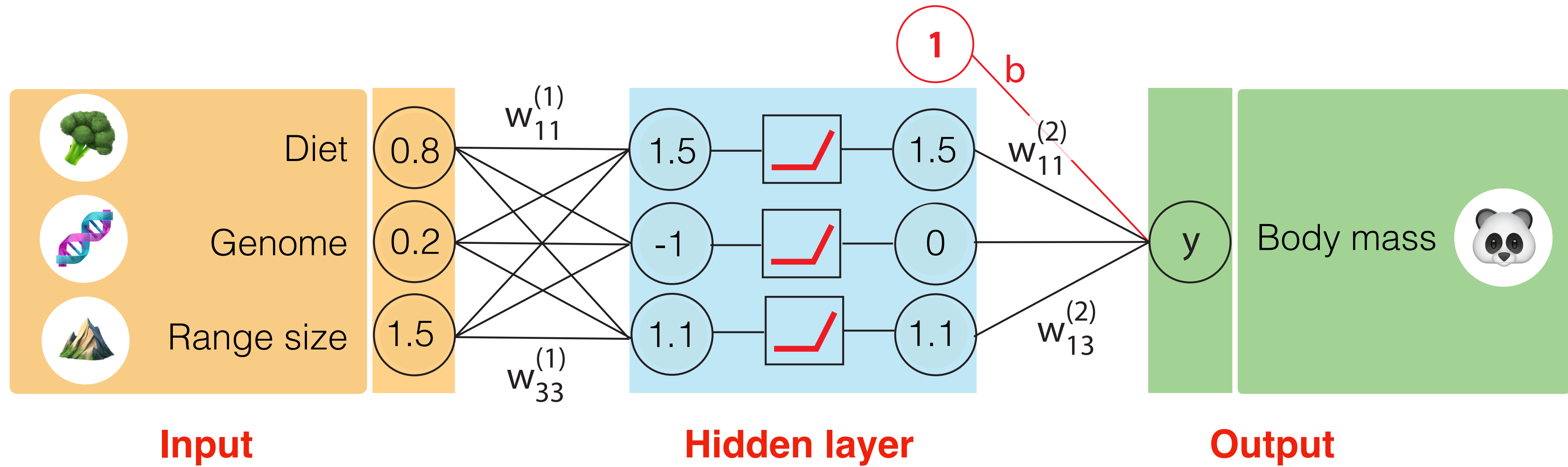


Bias node (cf. intercept)

$$\begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 1.1 \end{pmatrix} \begin{pmatrix} -0.4 & 0.1 & 2.0 & 2.6 \end{pmatrix} = \begin{pmatrix} -0.4 & 0.2 & 0 & 2.9 \end{pmatrix}$$

$z^{(2)}$ $w^{(1)}$

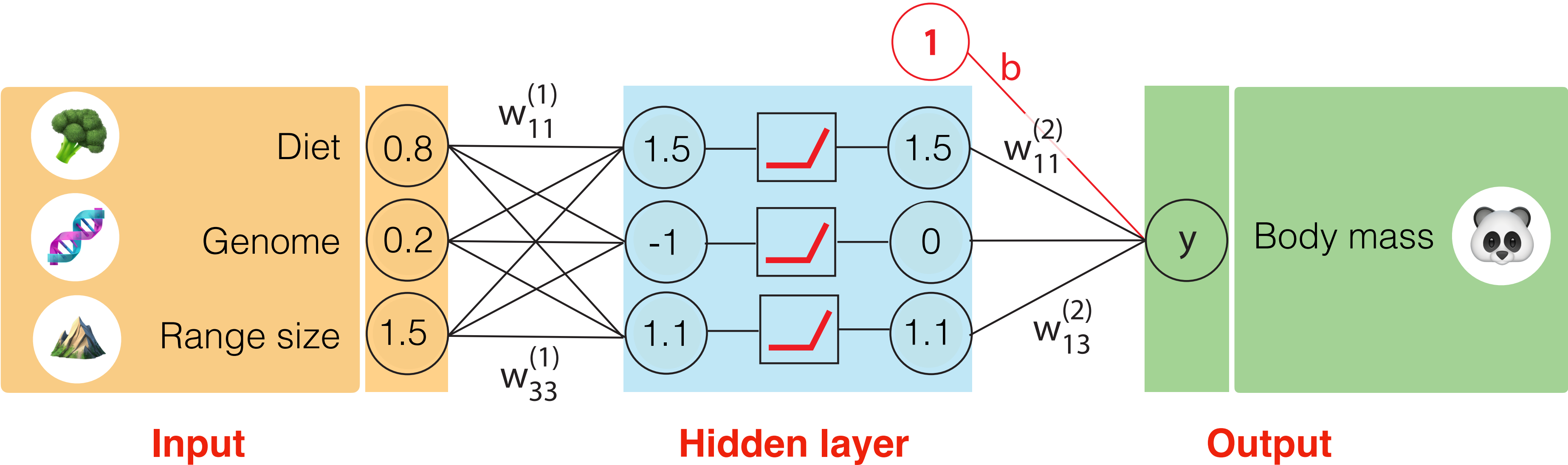
A neural network regression model



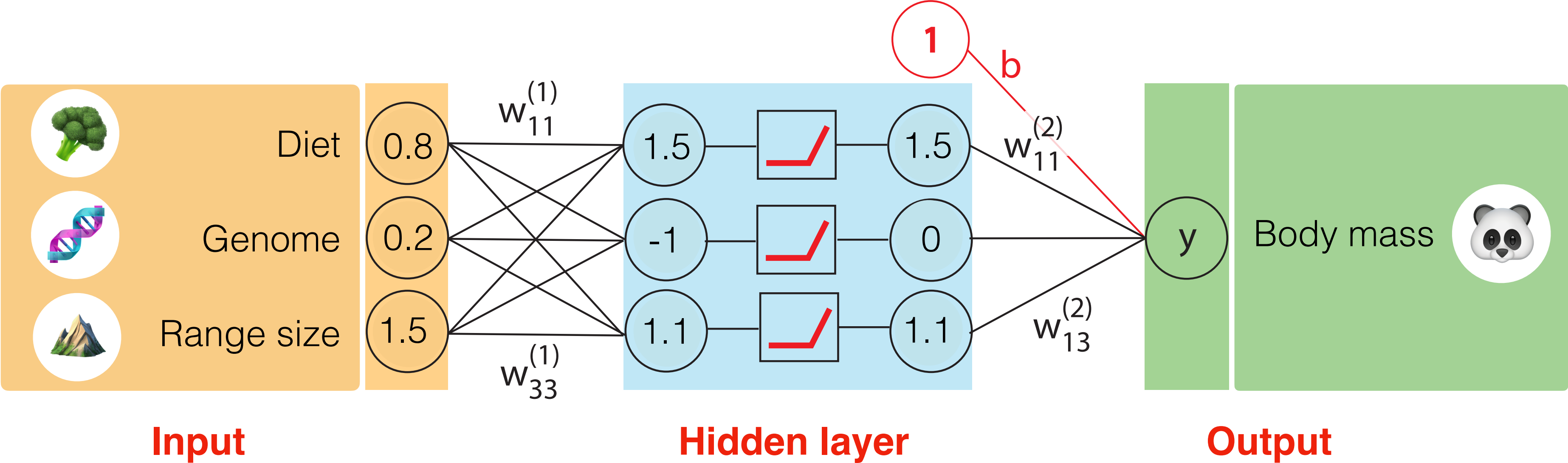
$$\begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 1.1 \end{pmatrix} (-0.4 \quad 0.1 \quad 2.0 \quad 2.6) = (-0.4 \quad 0.2 \quad 0 \quad 2.9)$$

$z^{(2)}$
 $w^{(1)}$

A neural network regression model



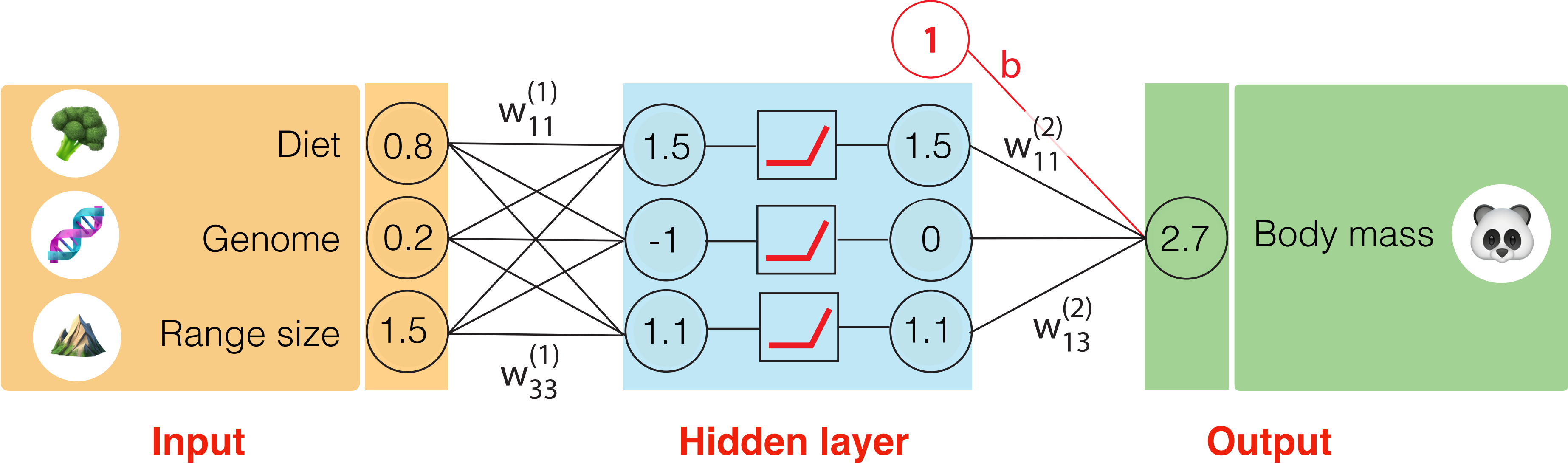
A neural network regression model



$$\begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 1.1 \end{pmatrix} \begin{pmatrix} -0.4 & 0.1 & 2.0 & 2.6 \end{pmatrix} = \begin{pmatrix} -0.4 & 0.2 & 0 & 2.9 \end{pmatrix}$$

$z^{(2)}$ $w^{(1)}$

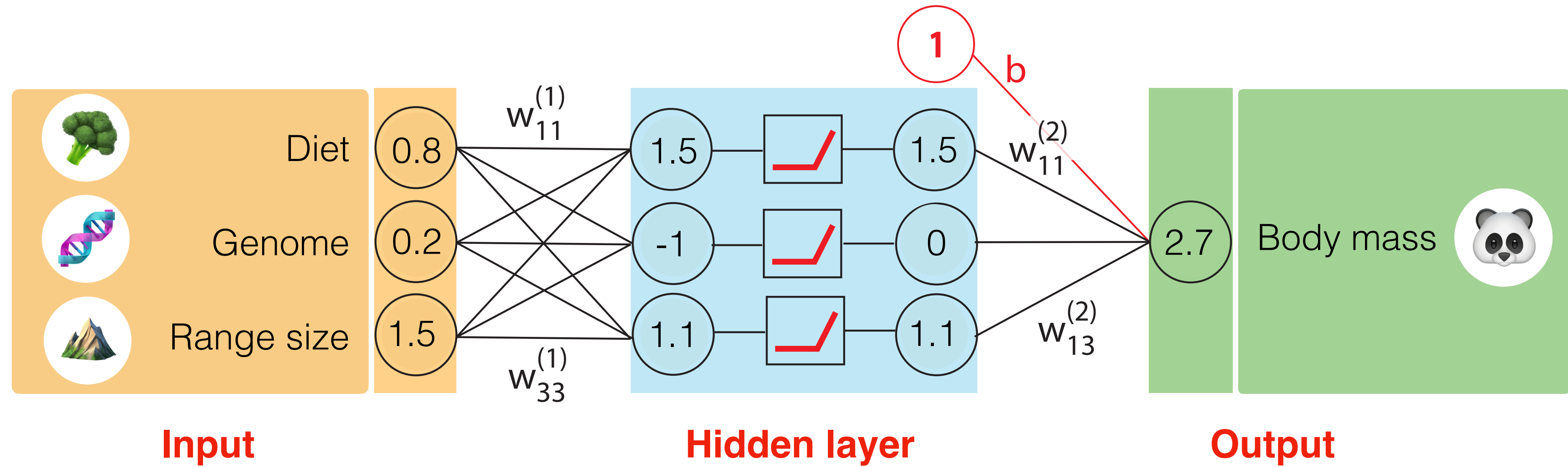
A neural network regression model



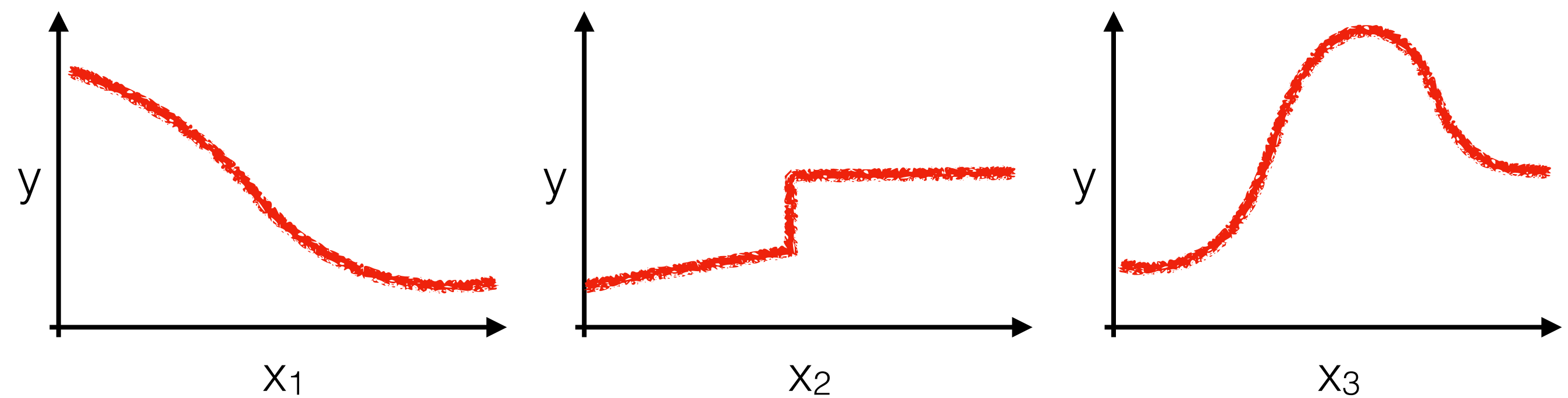
$$\begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 1.1 \end{pmatrix} \begin{pmatrix} -0.4 & 0.1 & 2.0 & 2.6 \end{pmatrix} = \boxed{\begin{pmatrix} -0.4 & 0.2 & 0 & 2.9 \end{pmatrix}} = 2.7$$

$z^{(2)}$ $w^{(1)}$ y

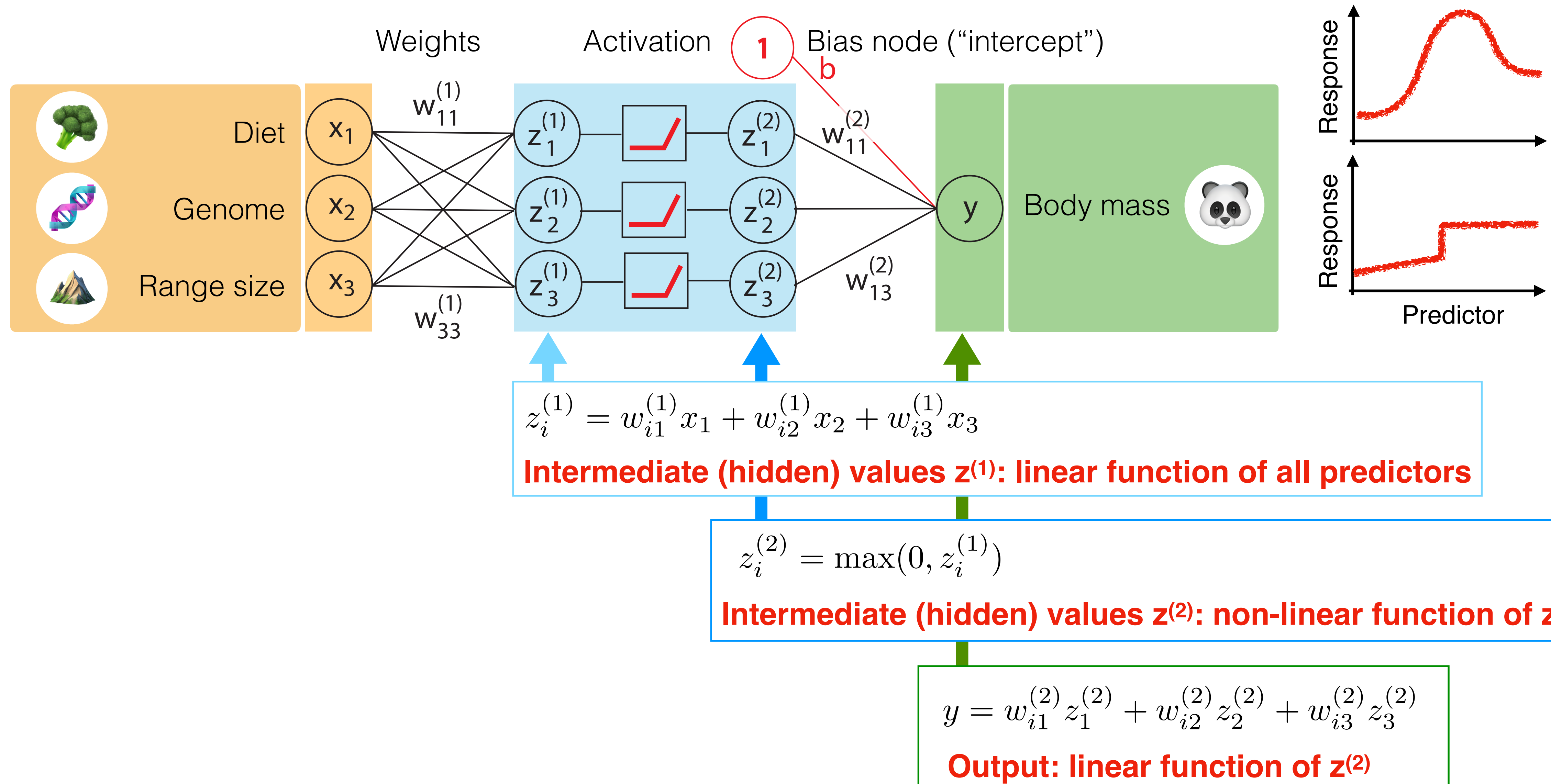
A neural network regression model



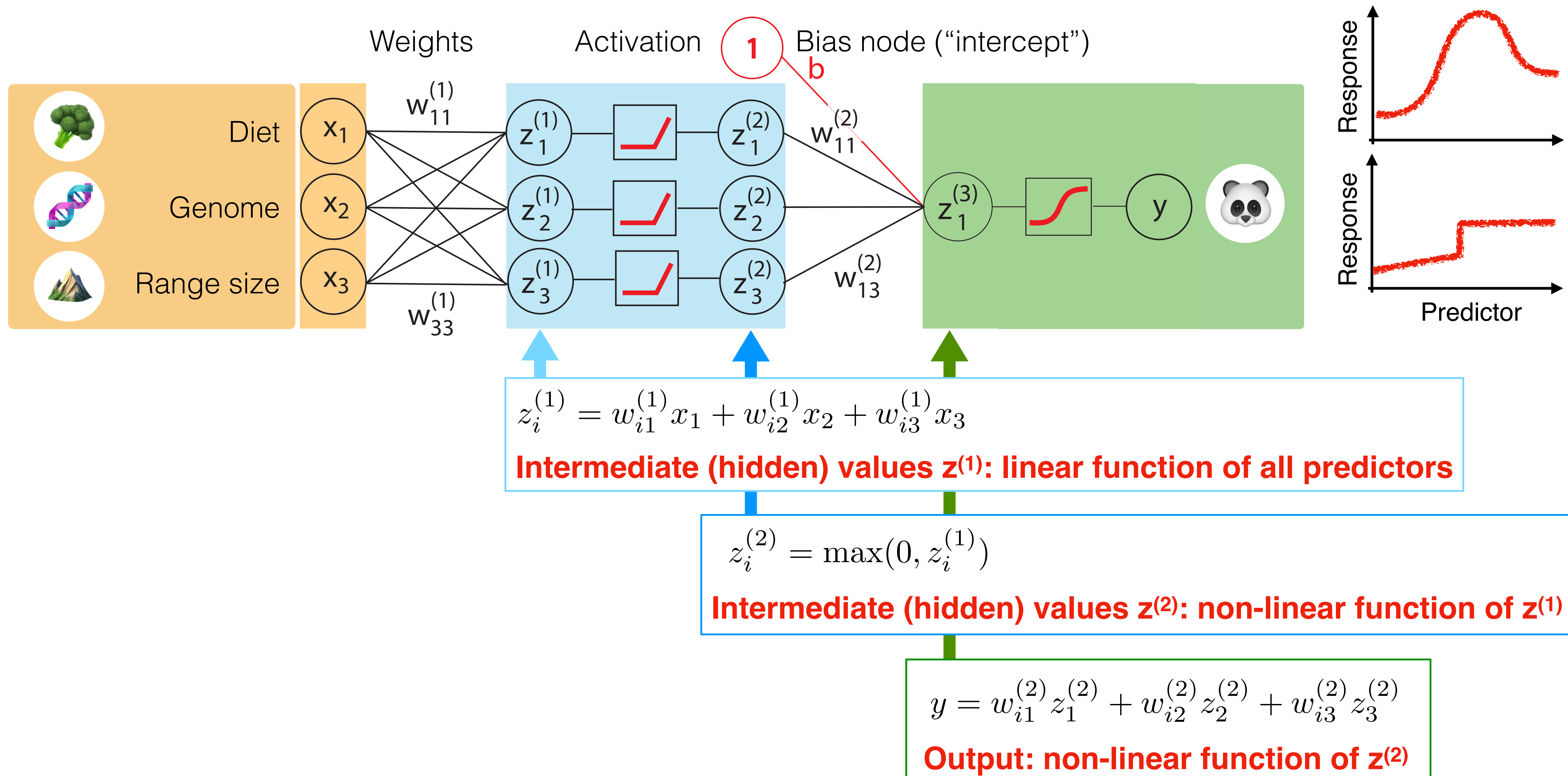
A neural network can approximate virtually any function



Neural network

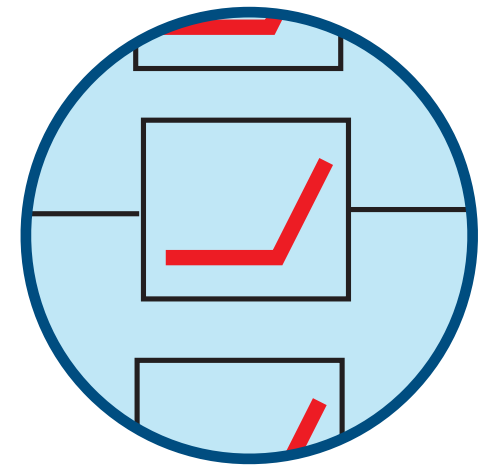


Neural network



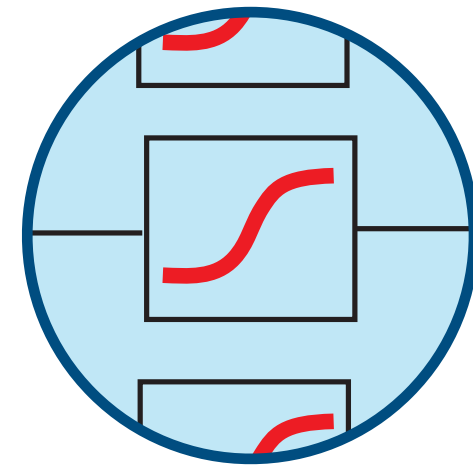
Parameterization of a neural network

Activation functions



ReLU: rectified linear unit

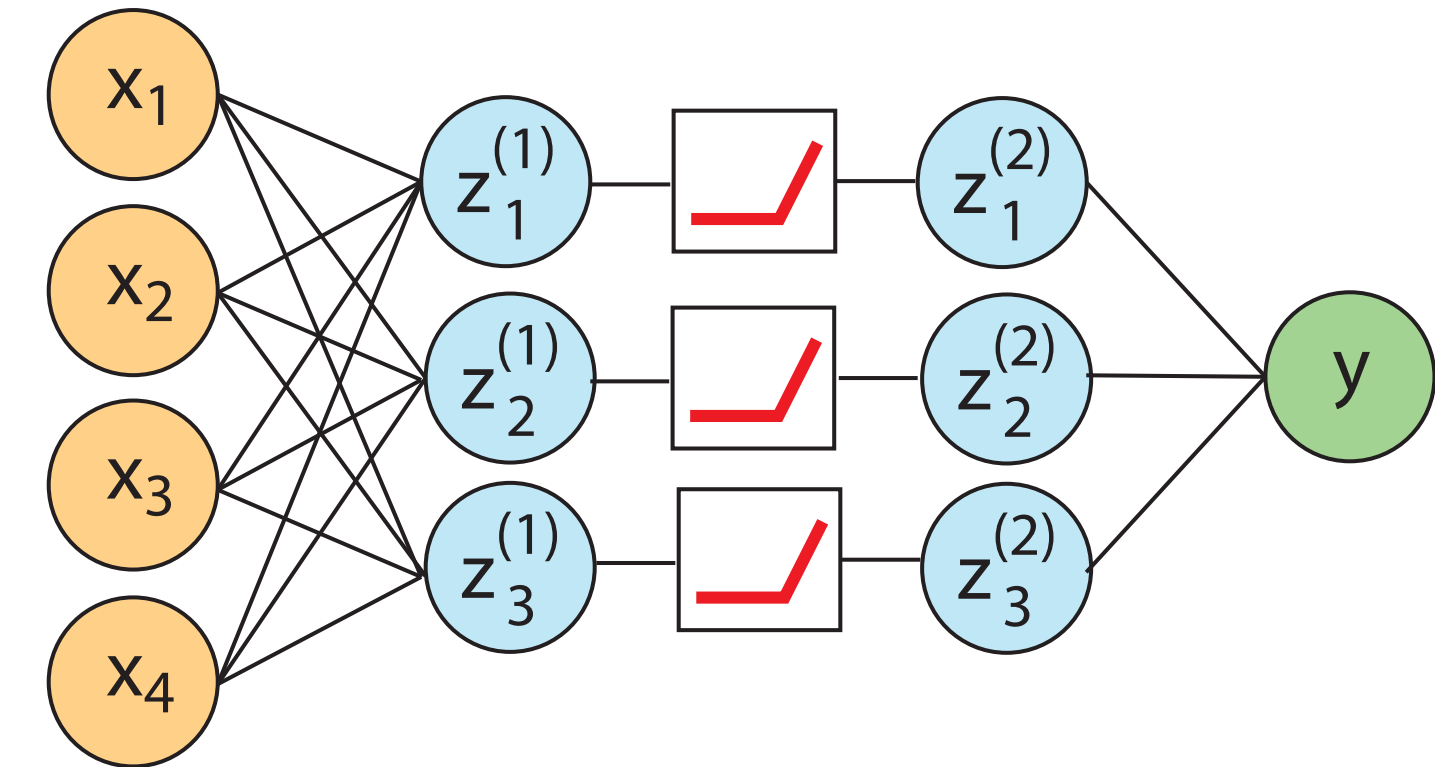
$$\text{ReLU}(x) = \max(0, x)$$



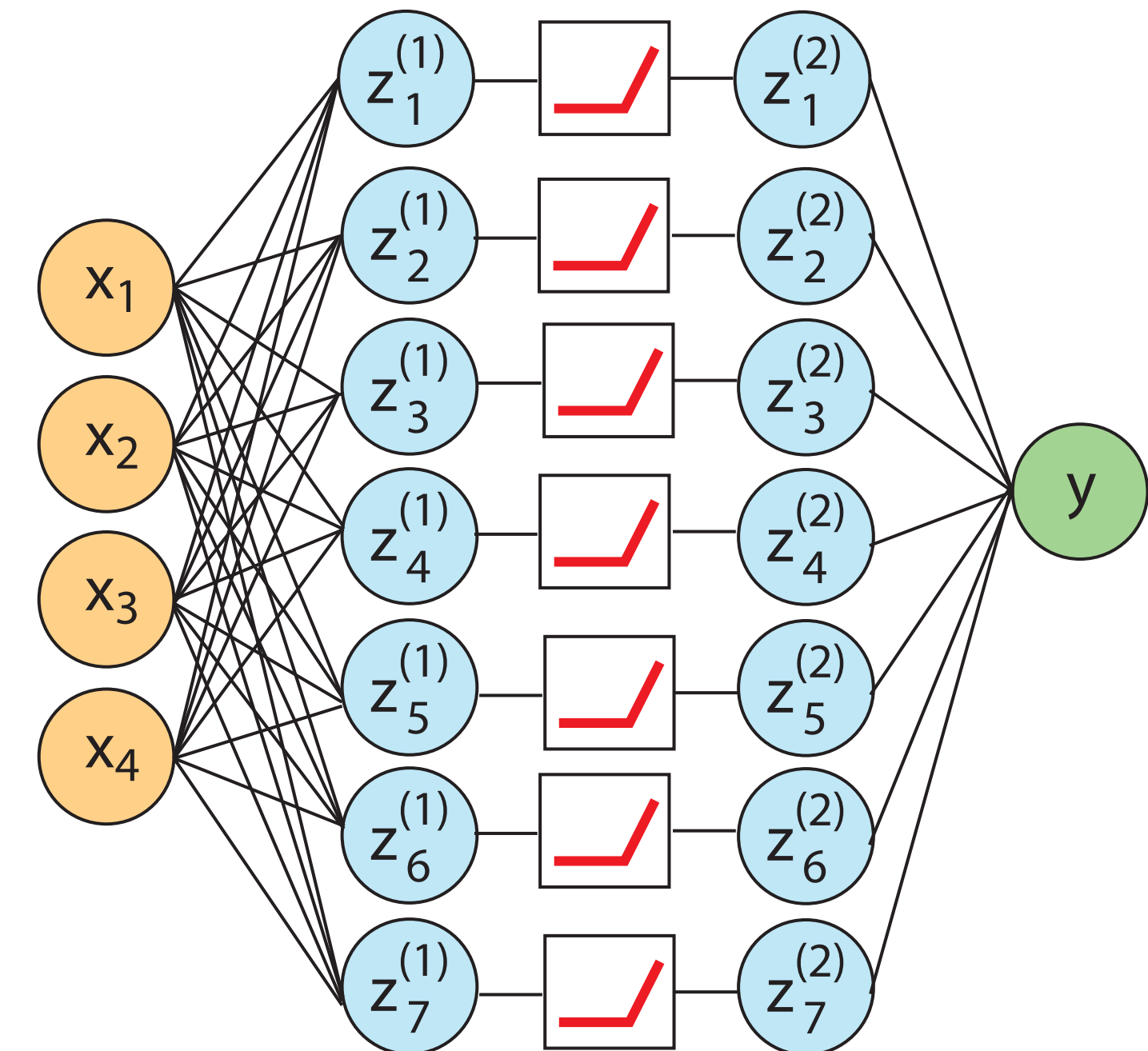
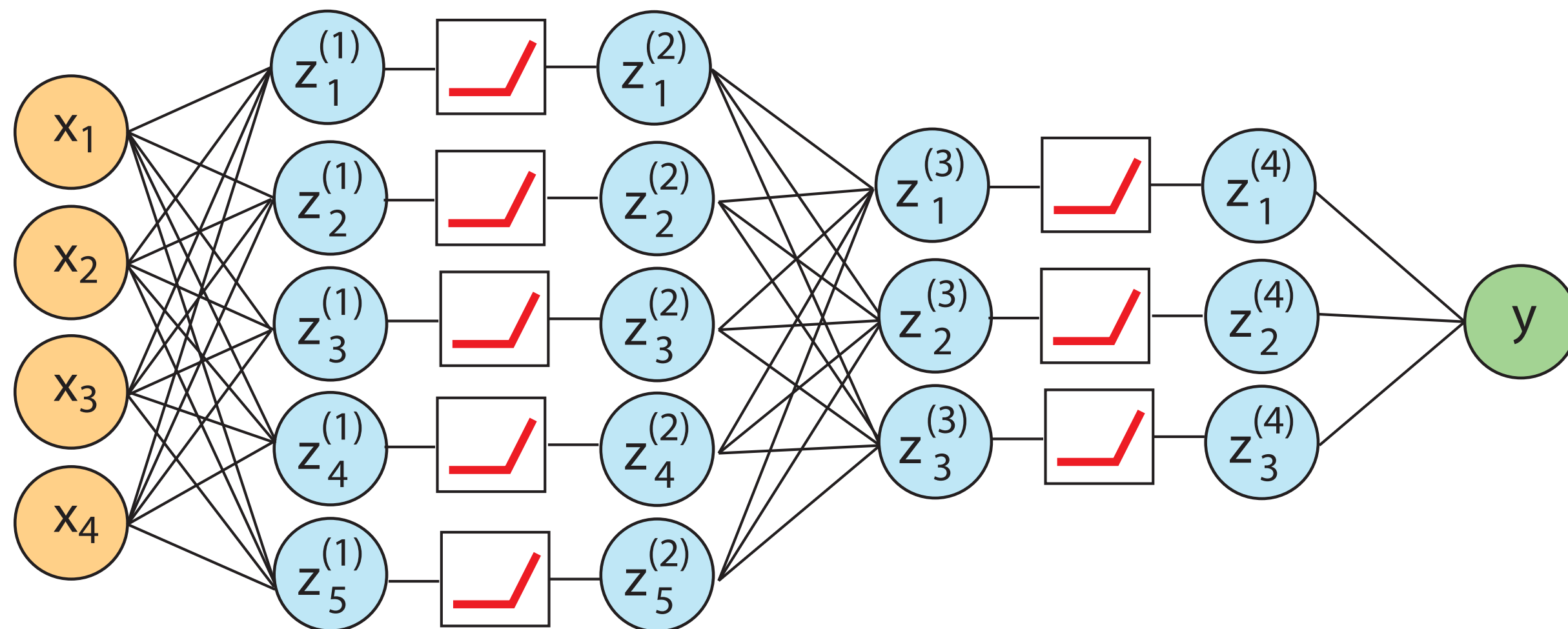
Sigmoid

$$\text{sigmoid}(x) = 1 / (1 + \exp(-x))$$

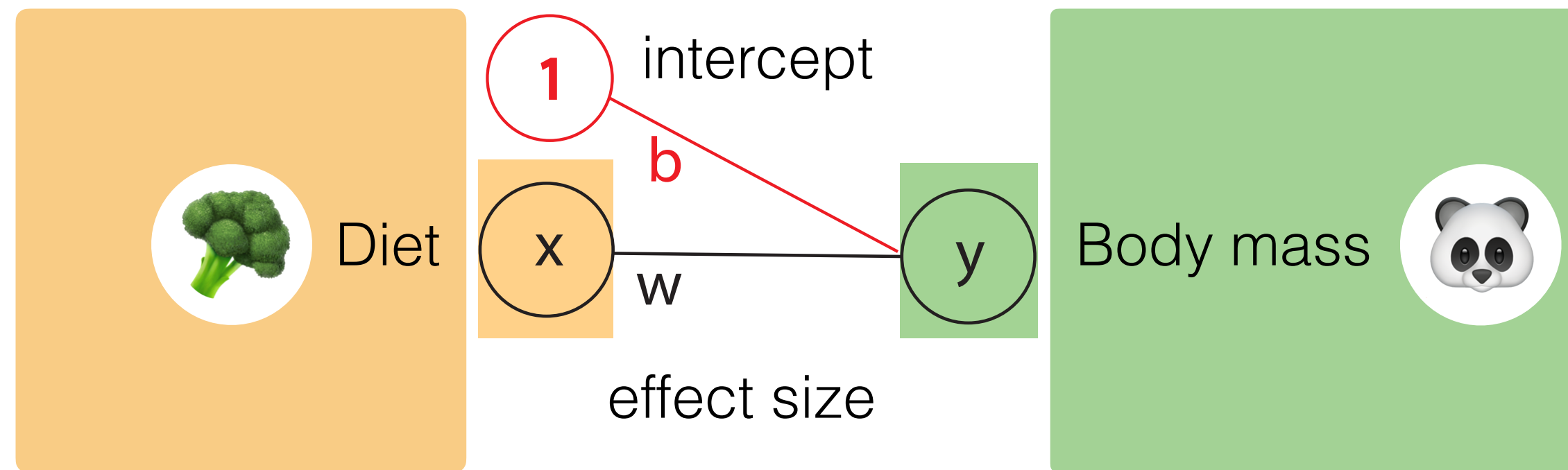
Number of nodes



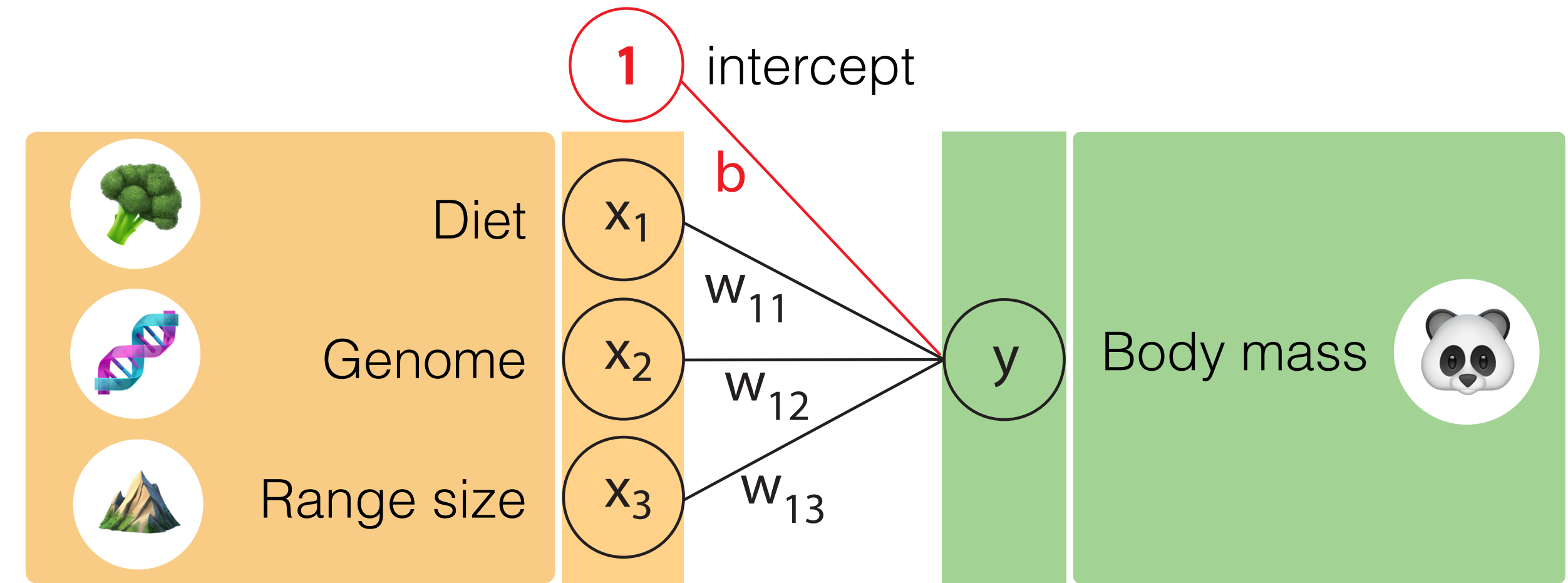
Number of hidden layers (deep NNs)



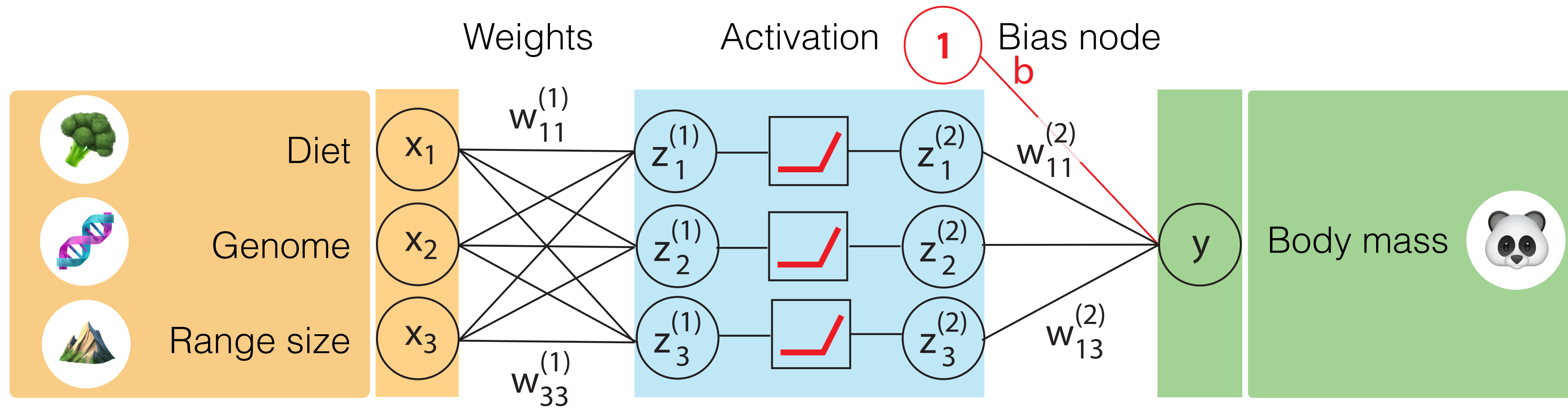
Linear regression



Multiple linear regression



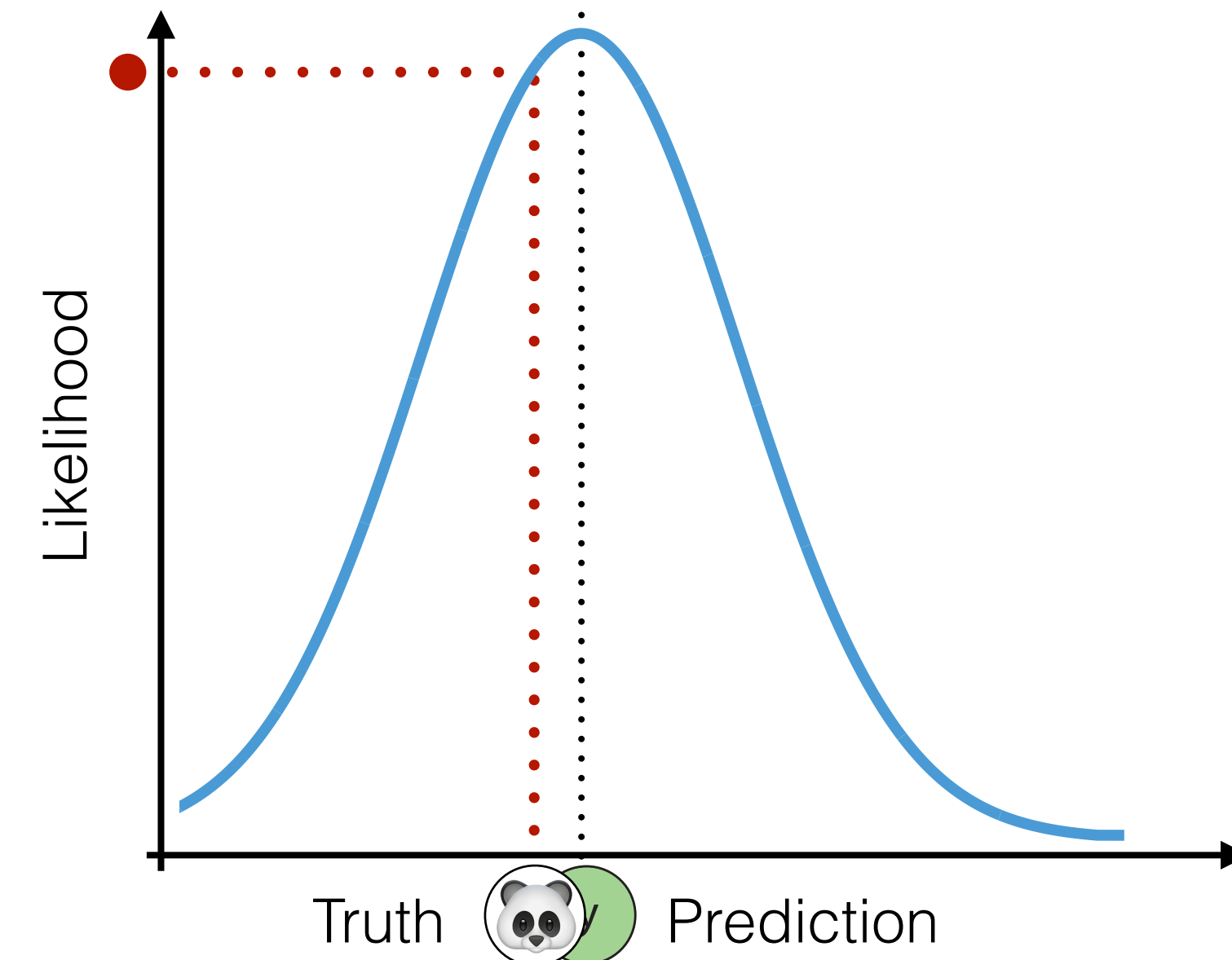
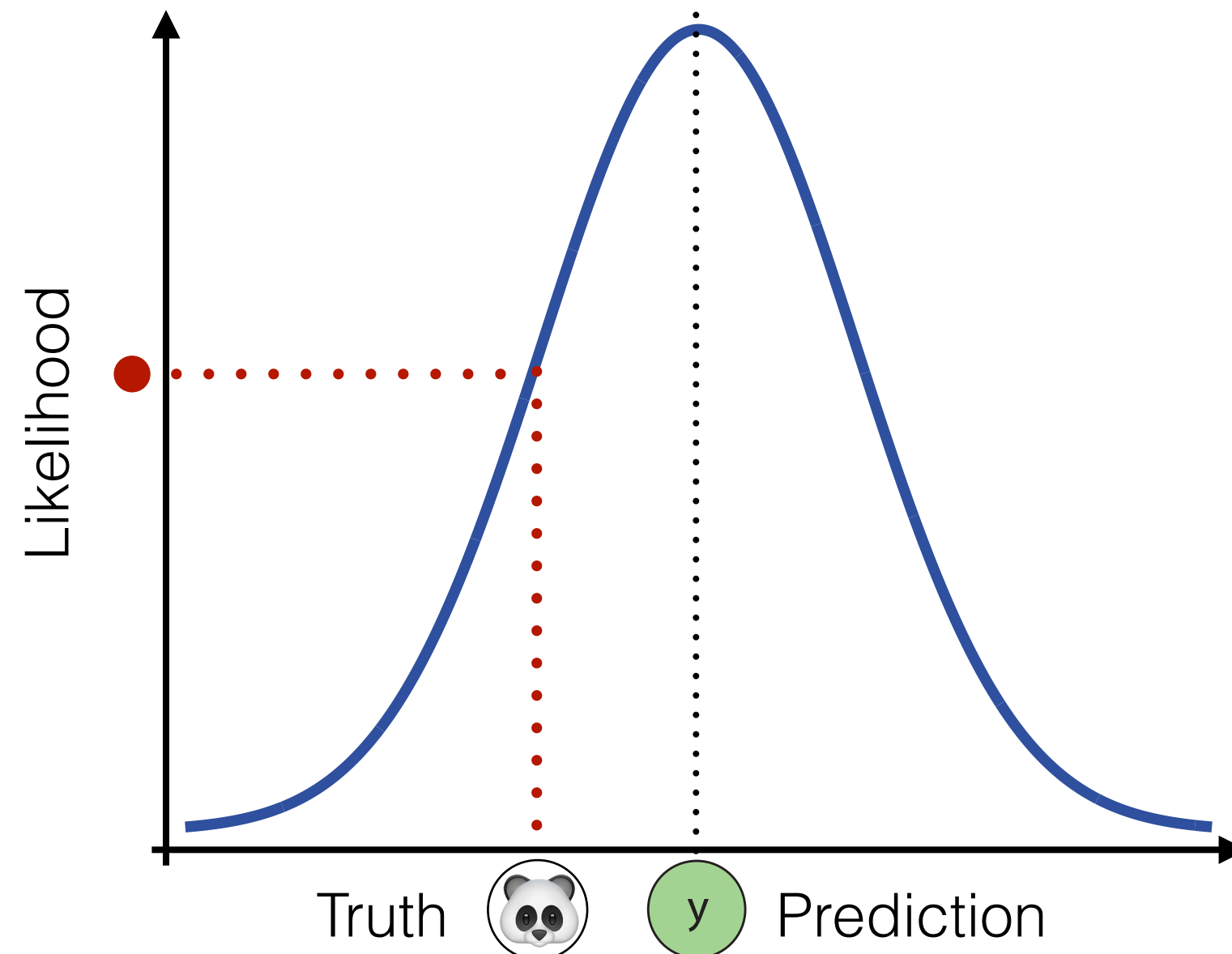
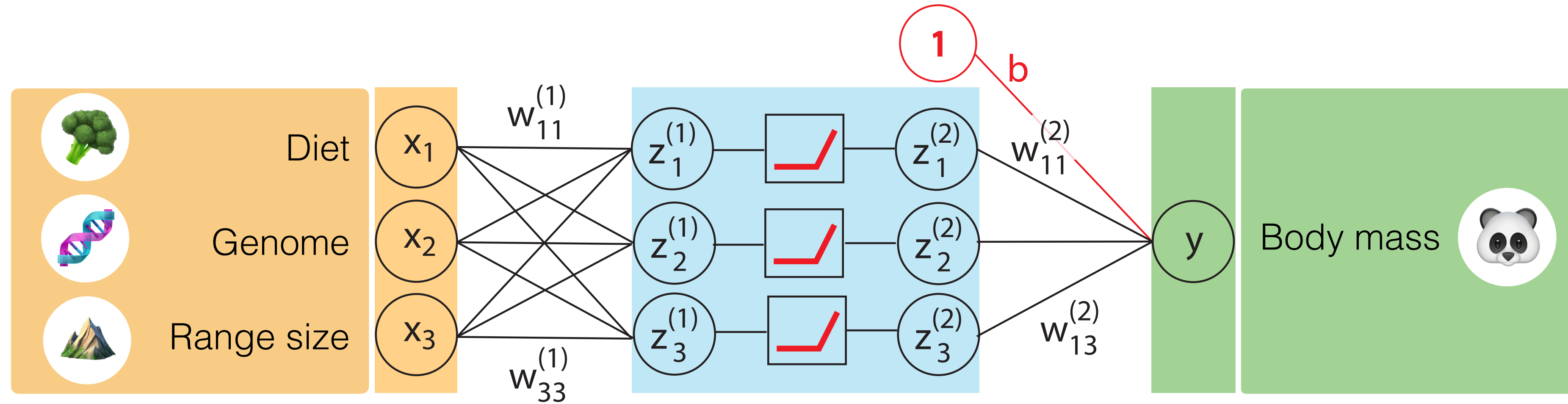
Neural network



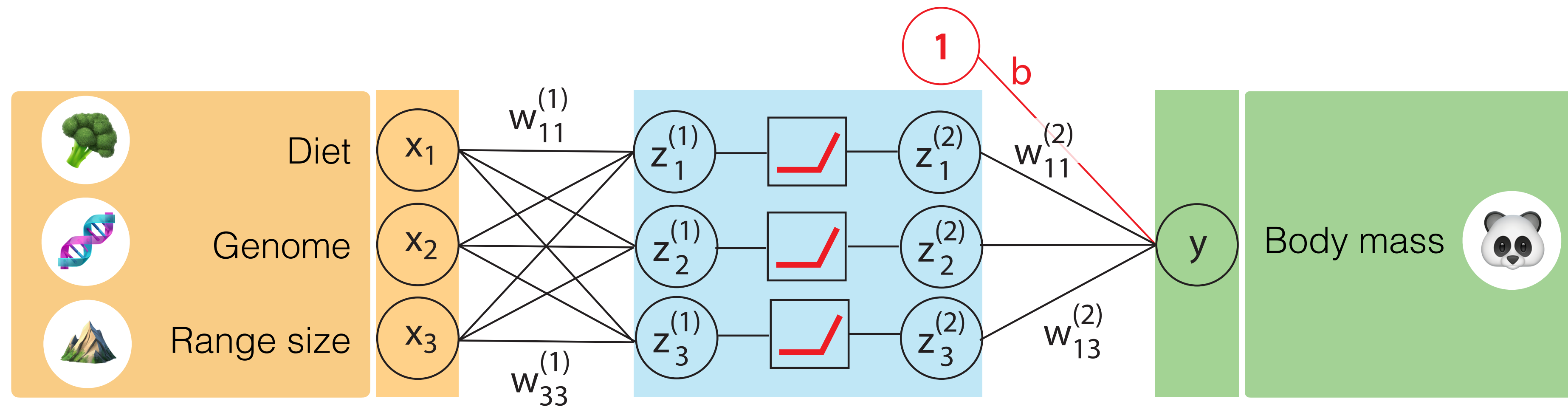
Pros and cons

- Non-linear responses
- Interaction among predictors
- No interpretable effect sizes
- Over-parameterized

Training a neural network: updating the parameter values to improve the predictions

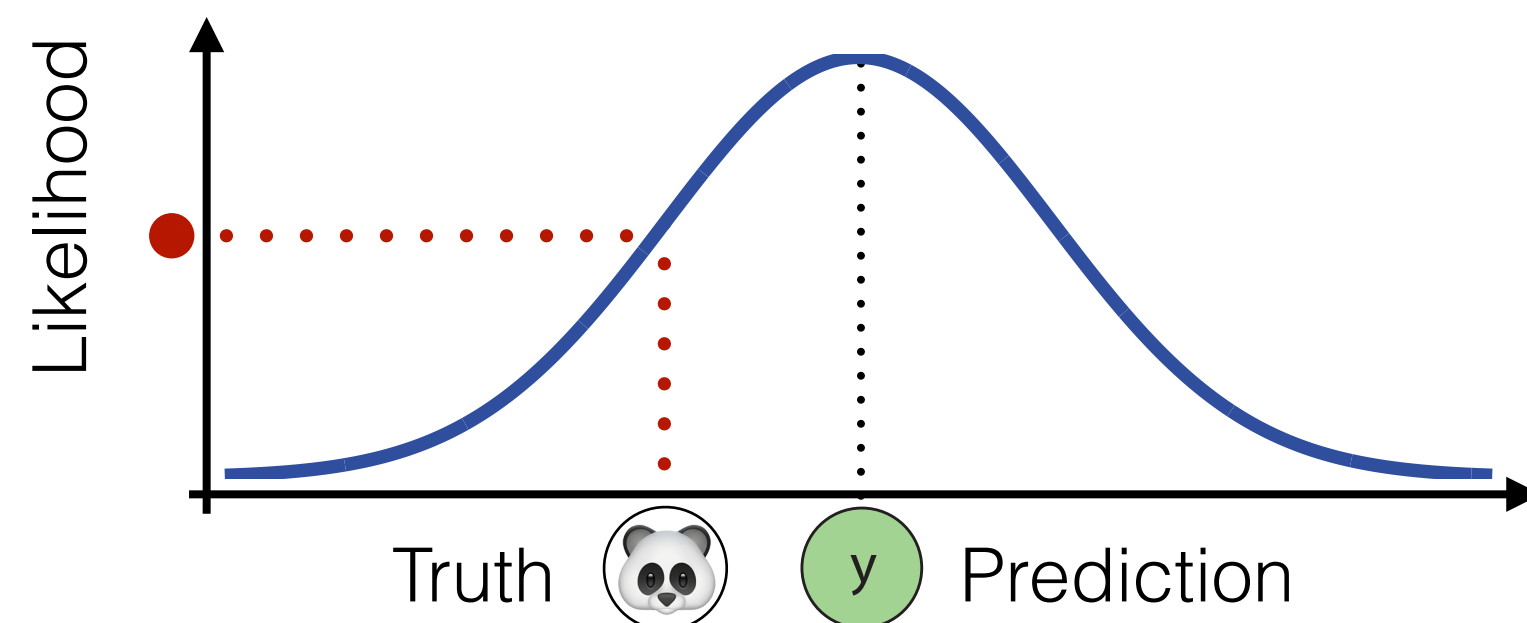


Training a BNN



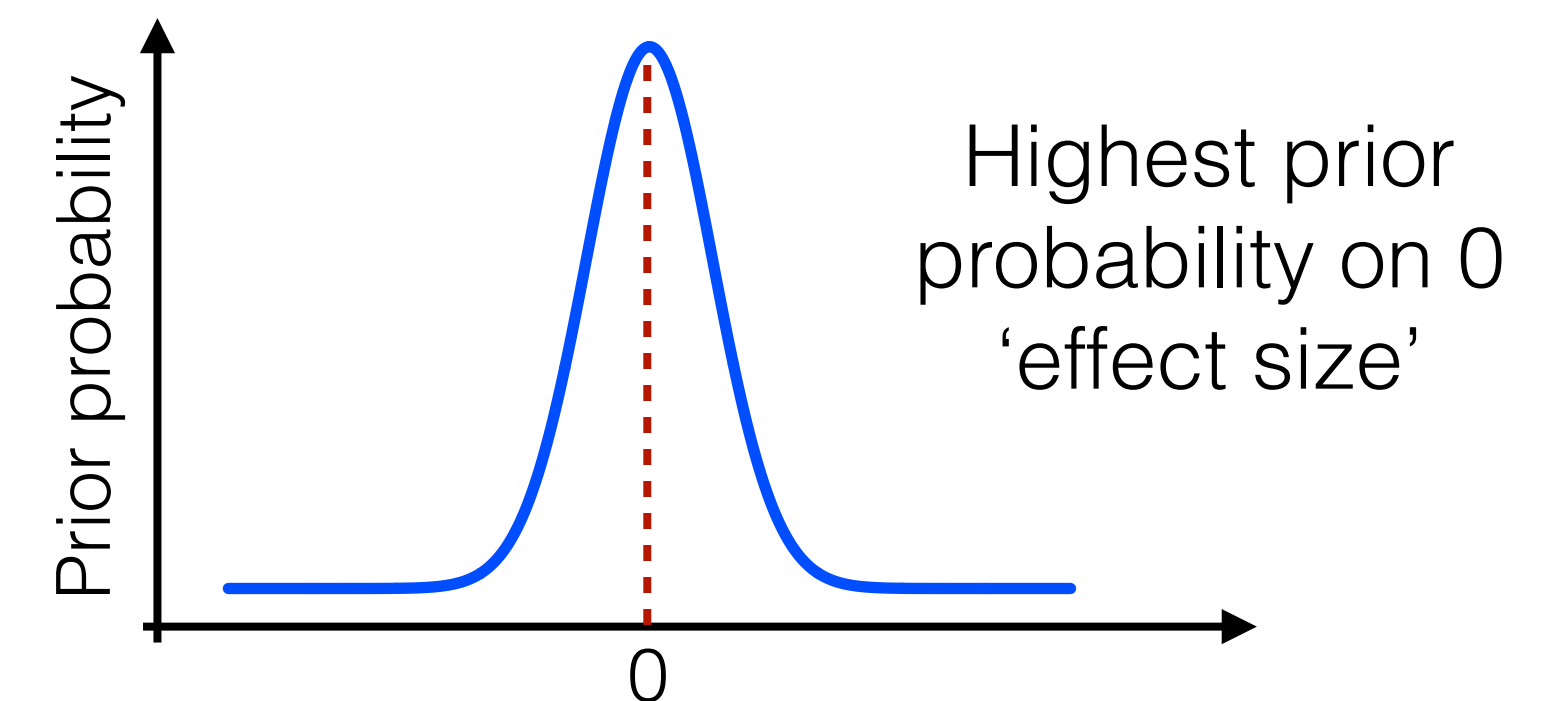
Likelihood function: the normal density function

$$P(\text{panda} | y, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{panda} - y)^2}{2\sigma^2}}$$



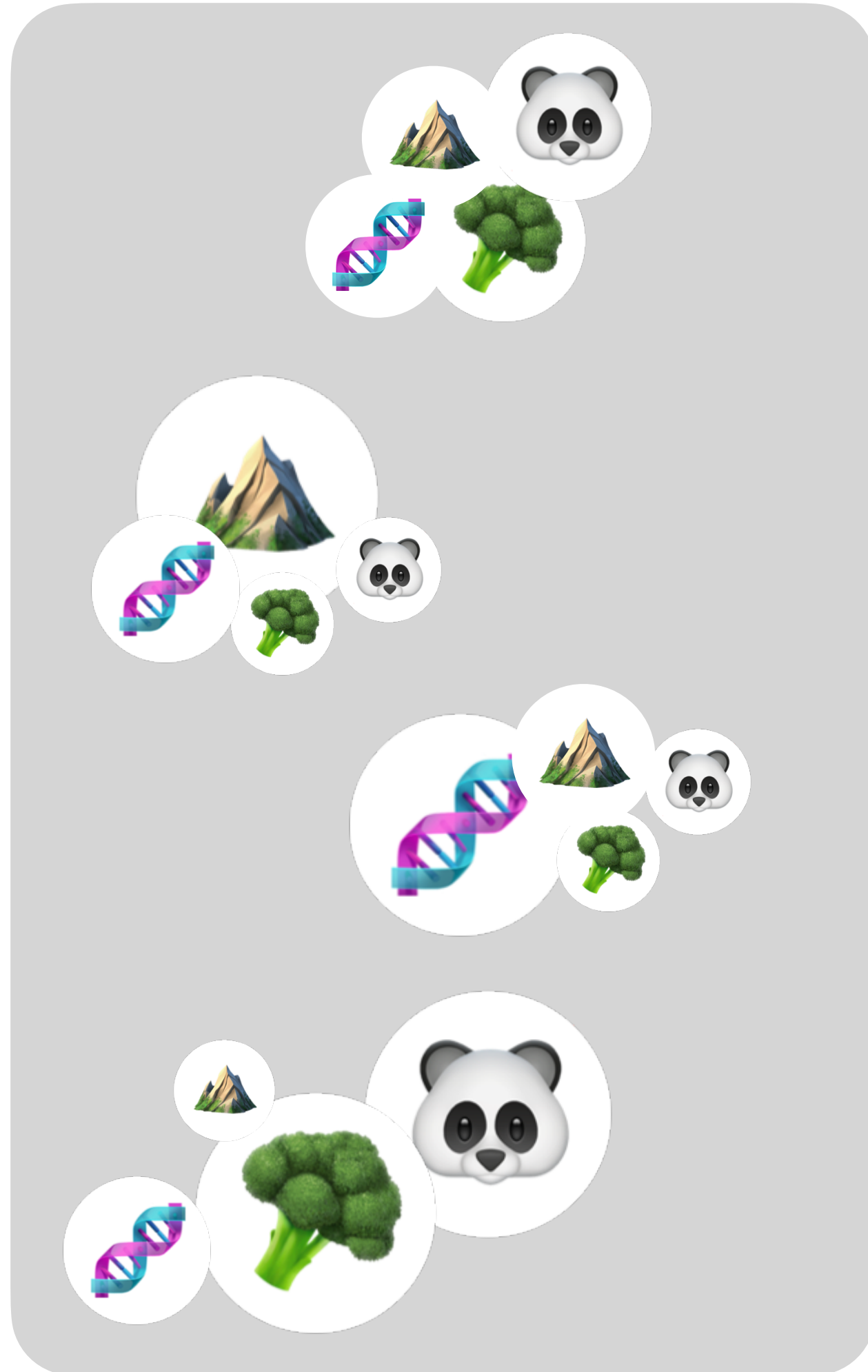
Priors on the weights

$$P(w) \sim \mathcal{N}(0, 1)$$

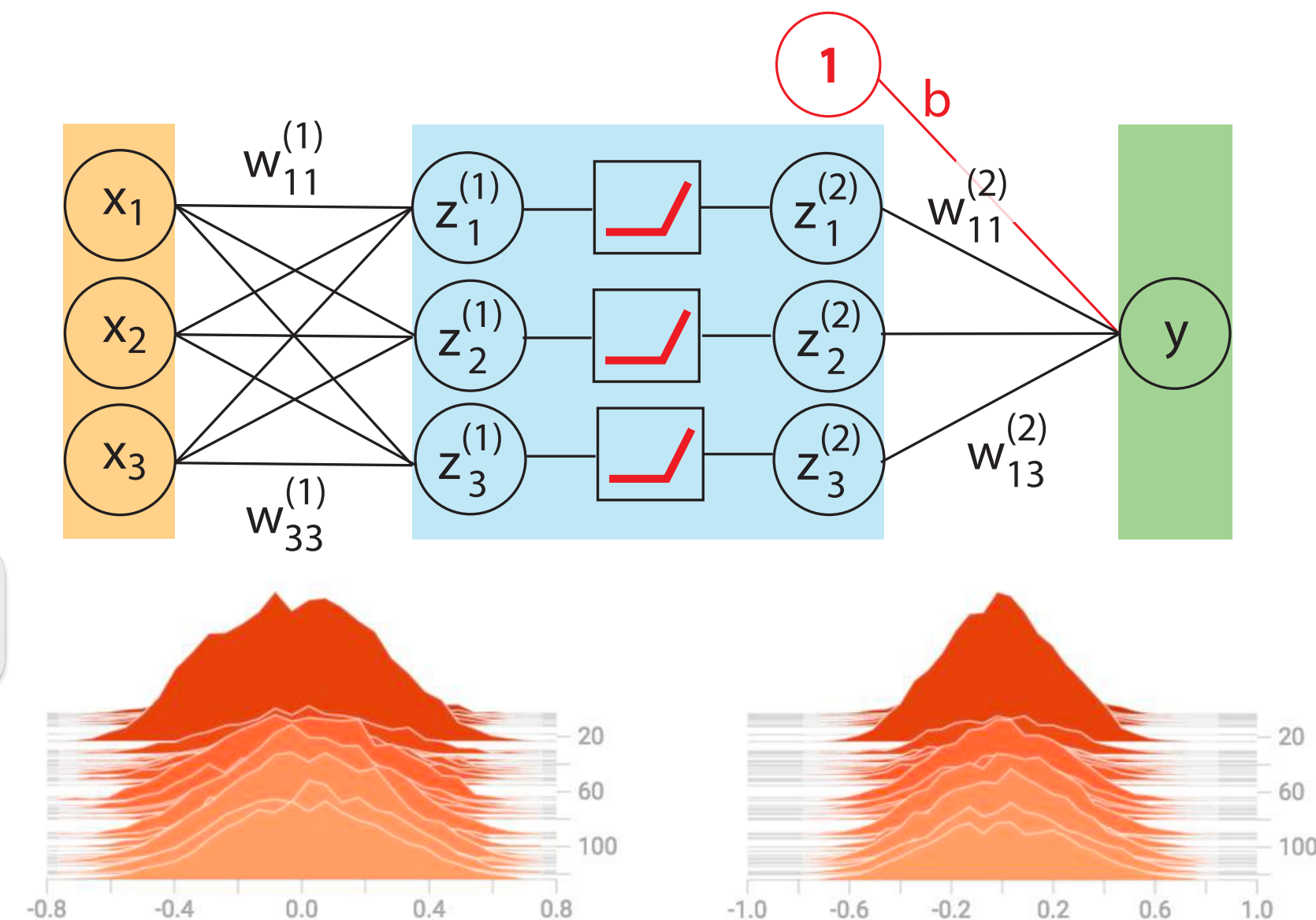
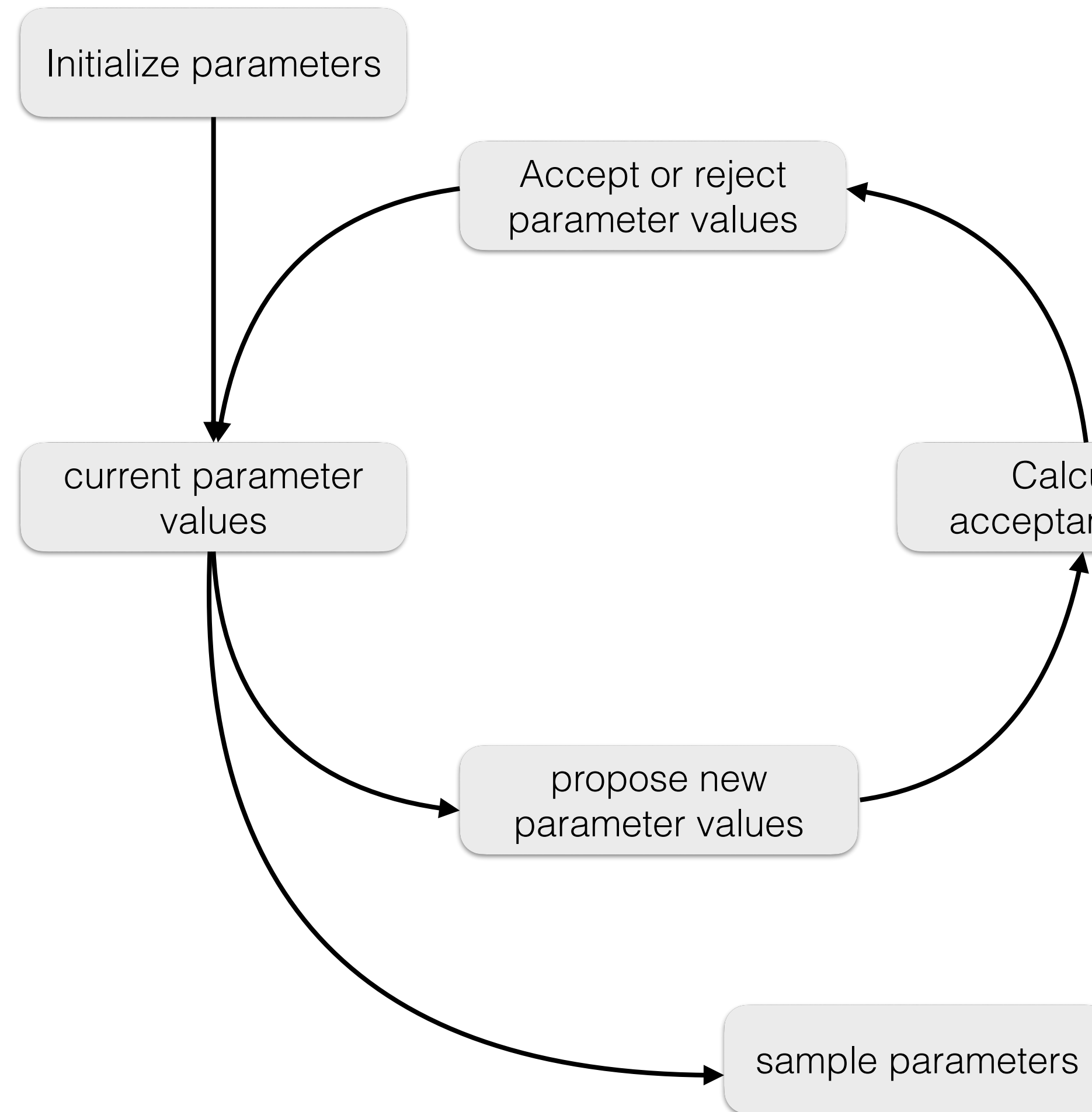


Training a BNN

Data ('training set')

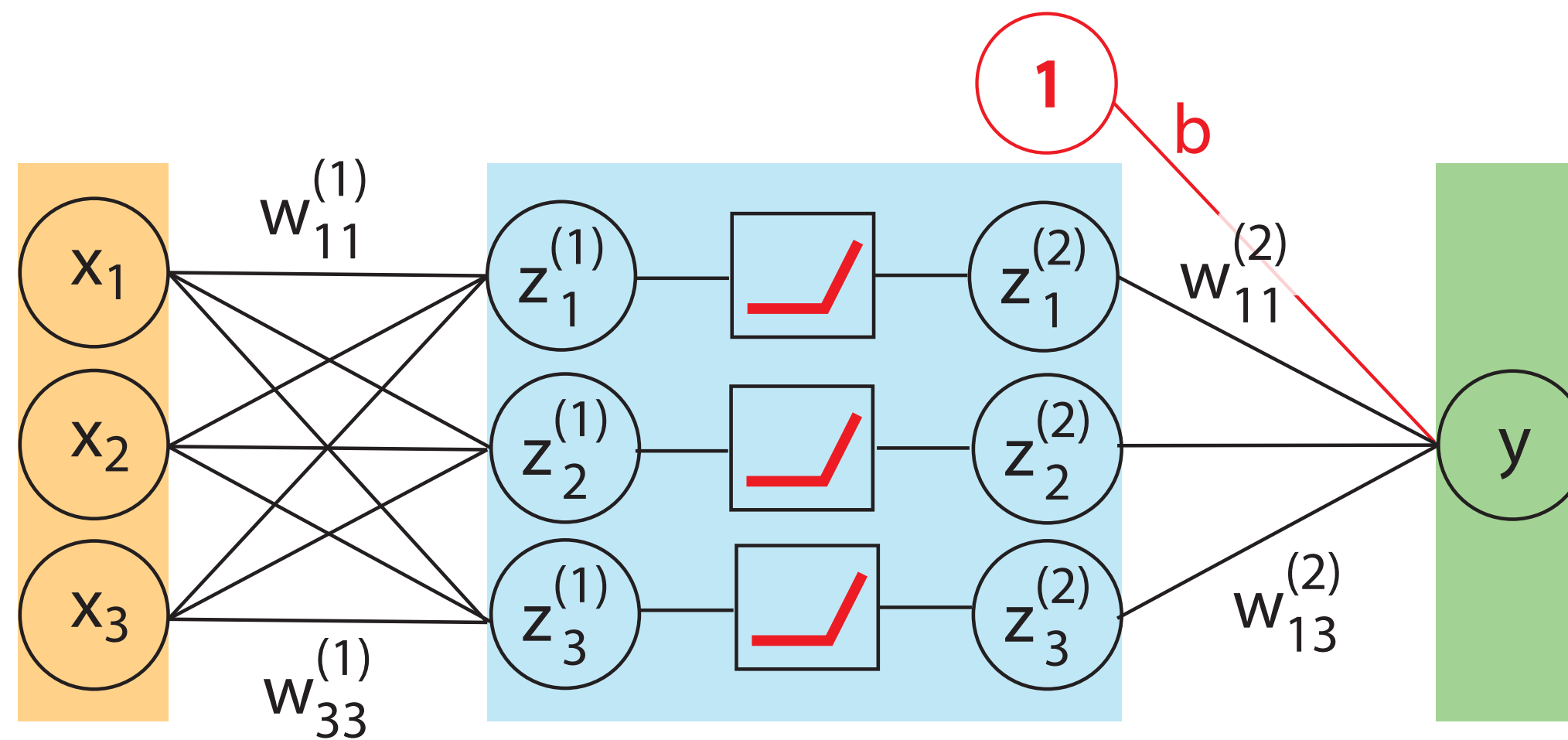
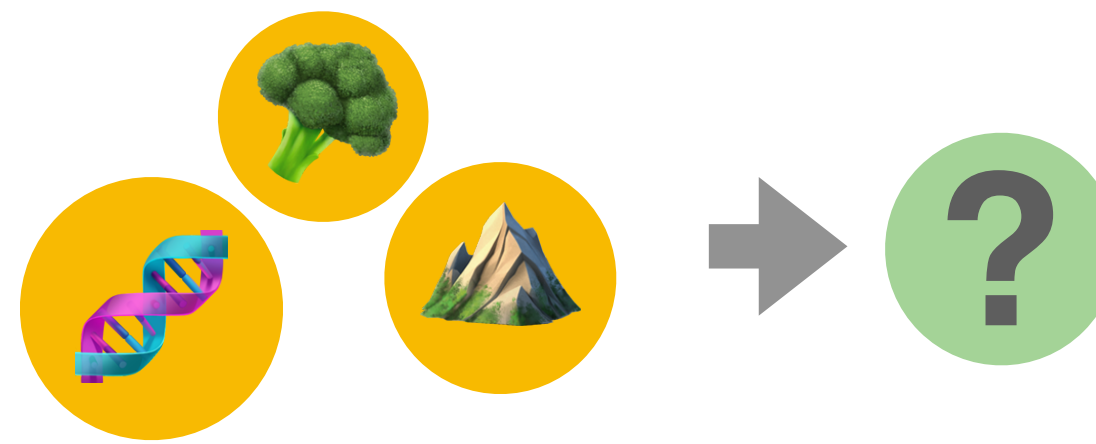


Posterior sampling algorithm, e.g. MCMC or HMC

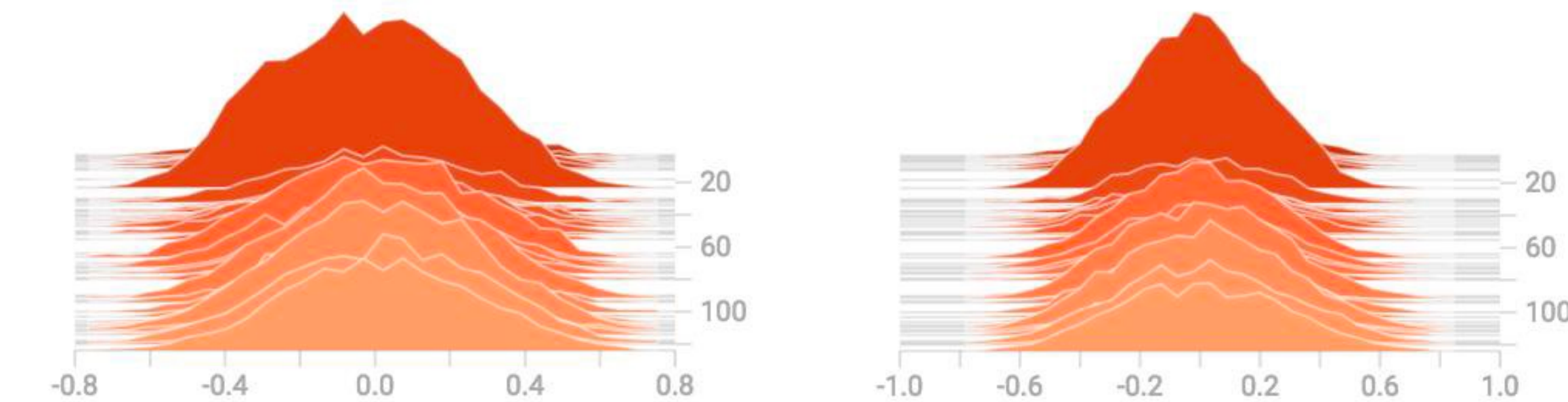


Predictions using a trained BNN

Predicting body mass from
diet, genome, range size

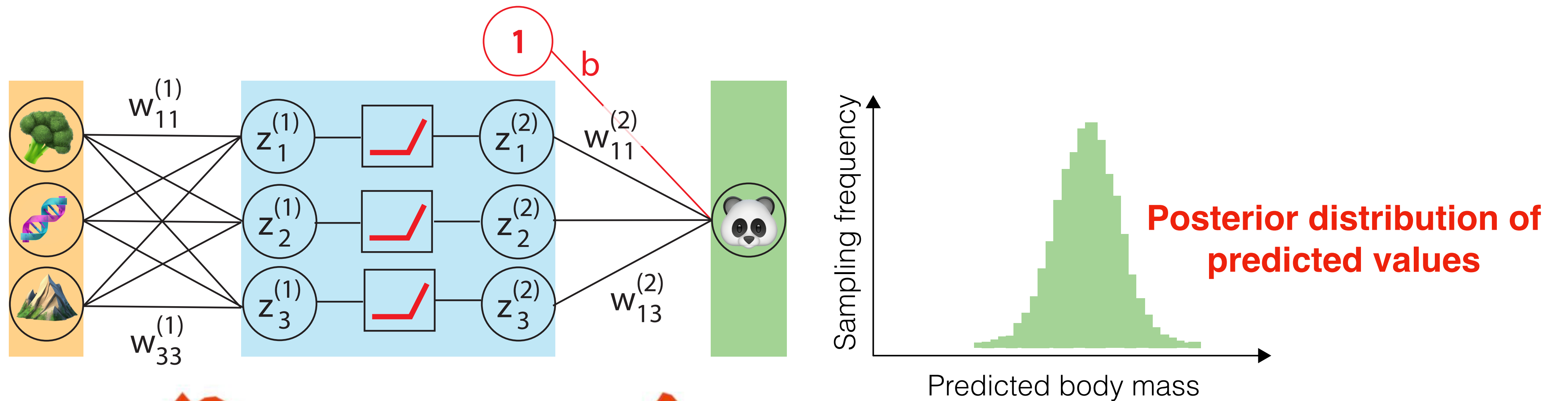
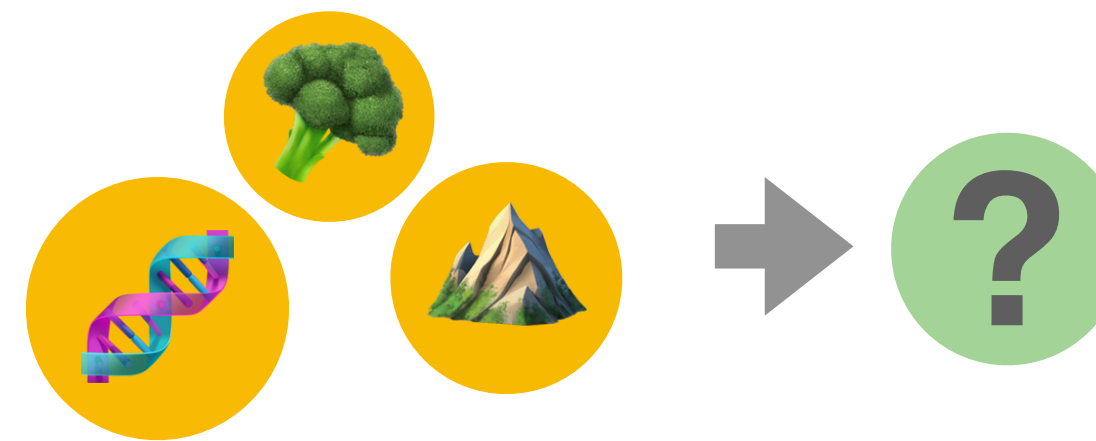


Posterior samples
of the weights

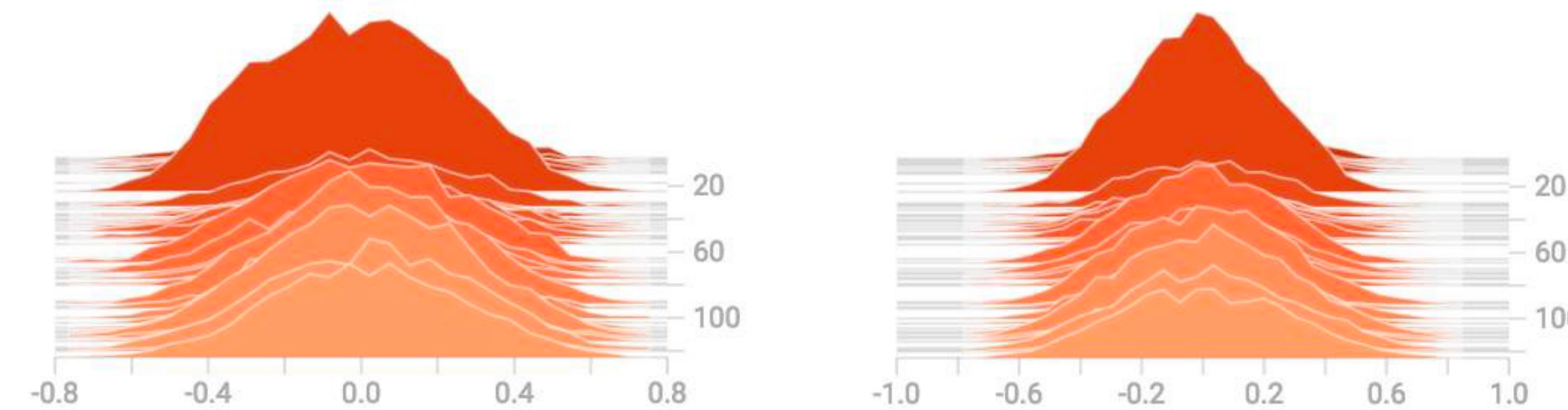


Predictions using a trained BNN

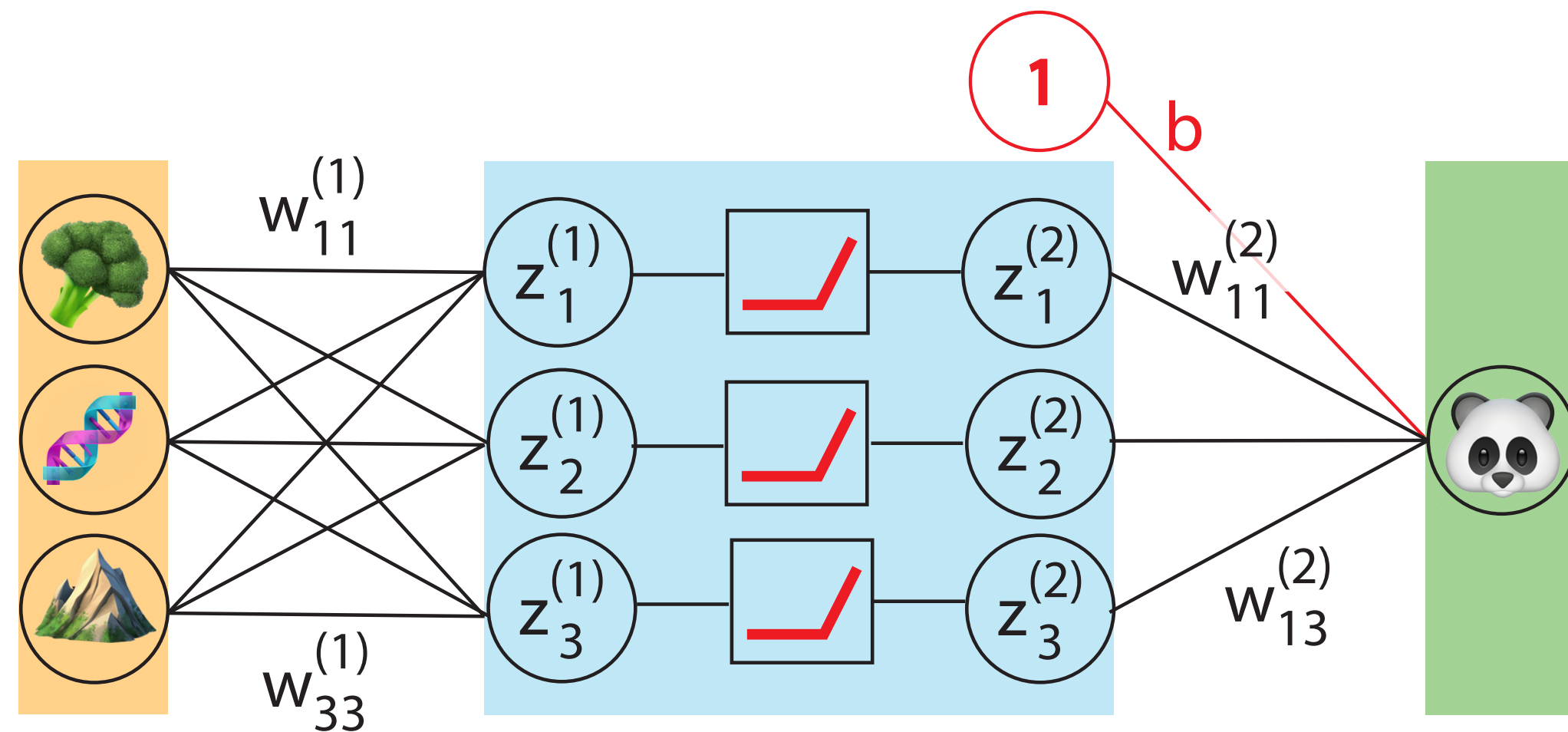
Predicting body mass from
diet, genome, range size



Posterior samples
of the weights

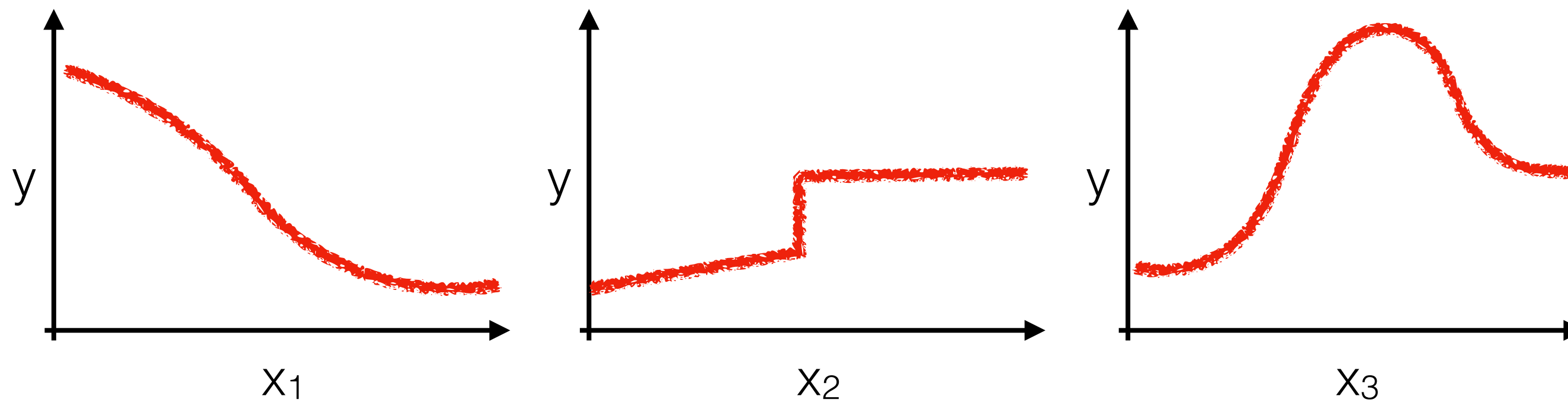


Neural networks can approximate virtually any function



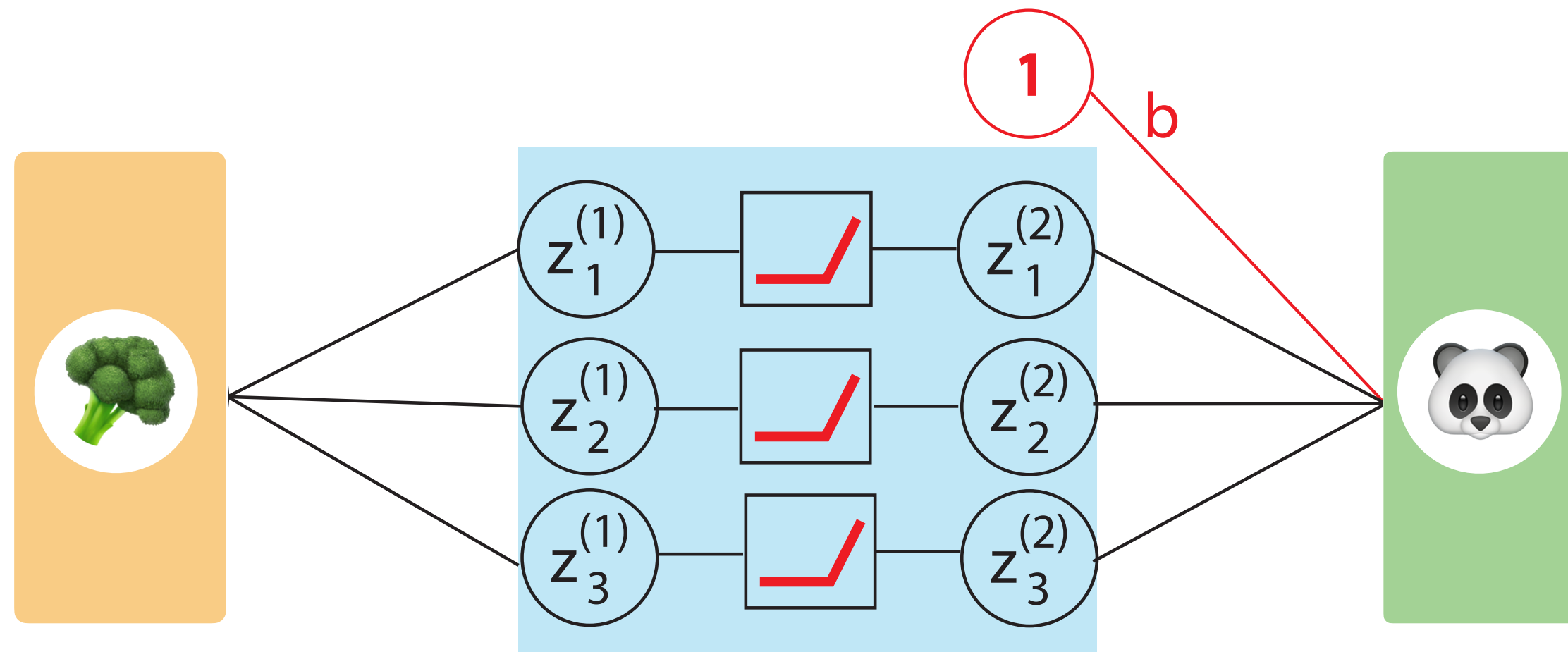
BNNs are powerful but...

- They are over-parameterized models (risk of overfitting, potential issues in extrapolation)
- Defining informative priors is not trivial (if at all possible)
- Estimated parameters are not directly interpretable
- They are not designed for hypothesis testing



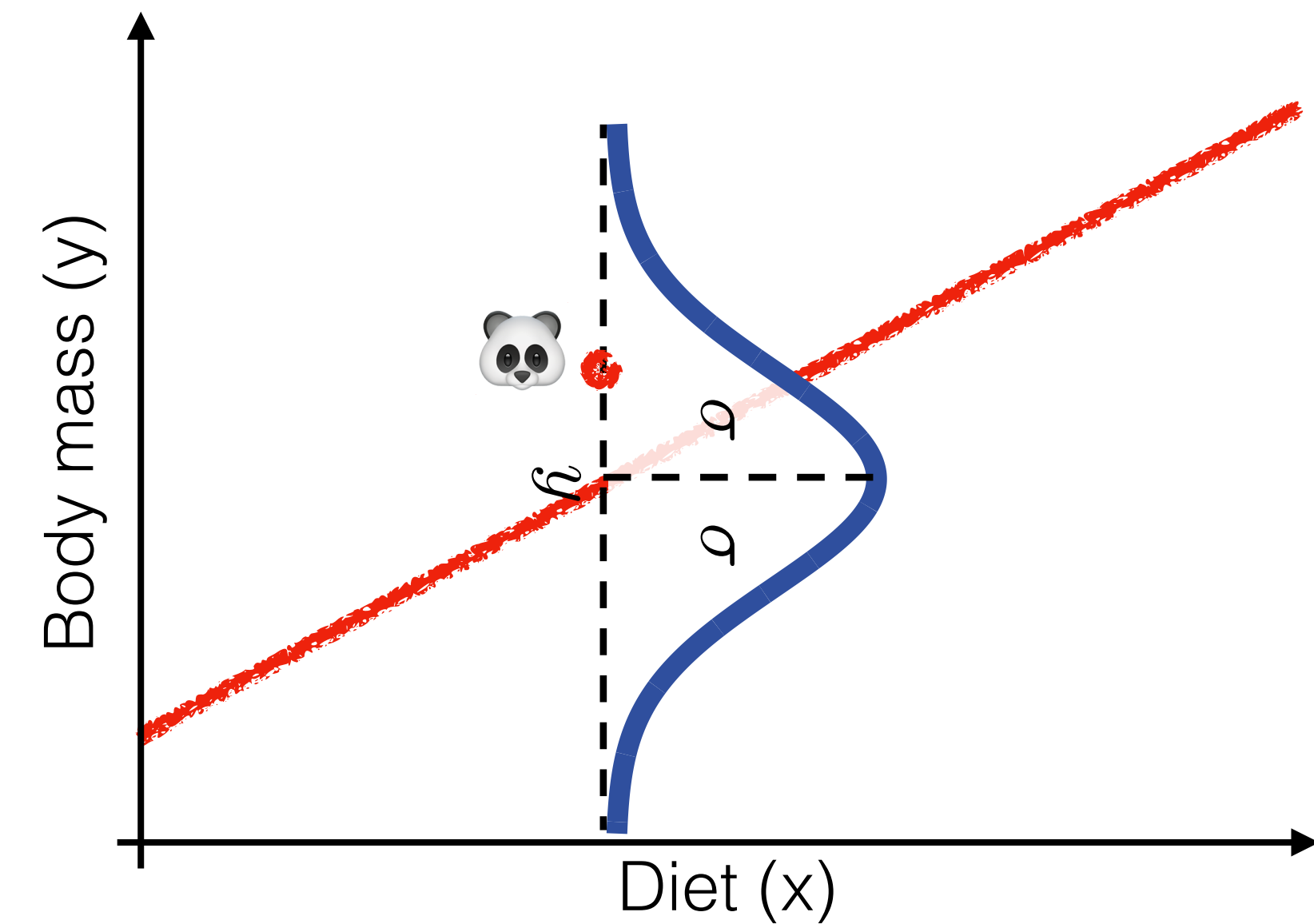
Training a supervised neural network

NN regression



Likelihood of an observed panda body mass

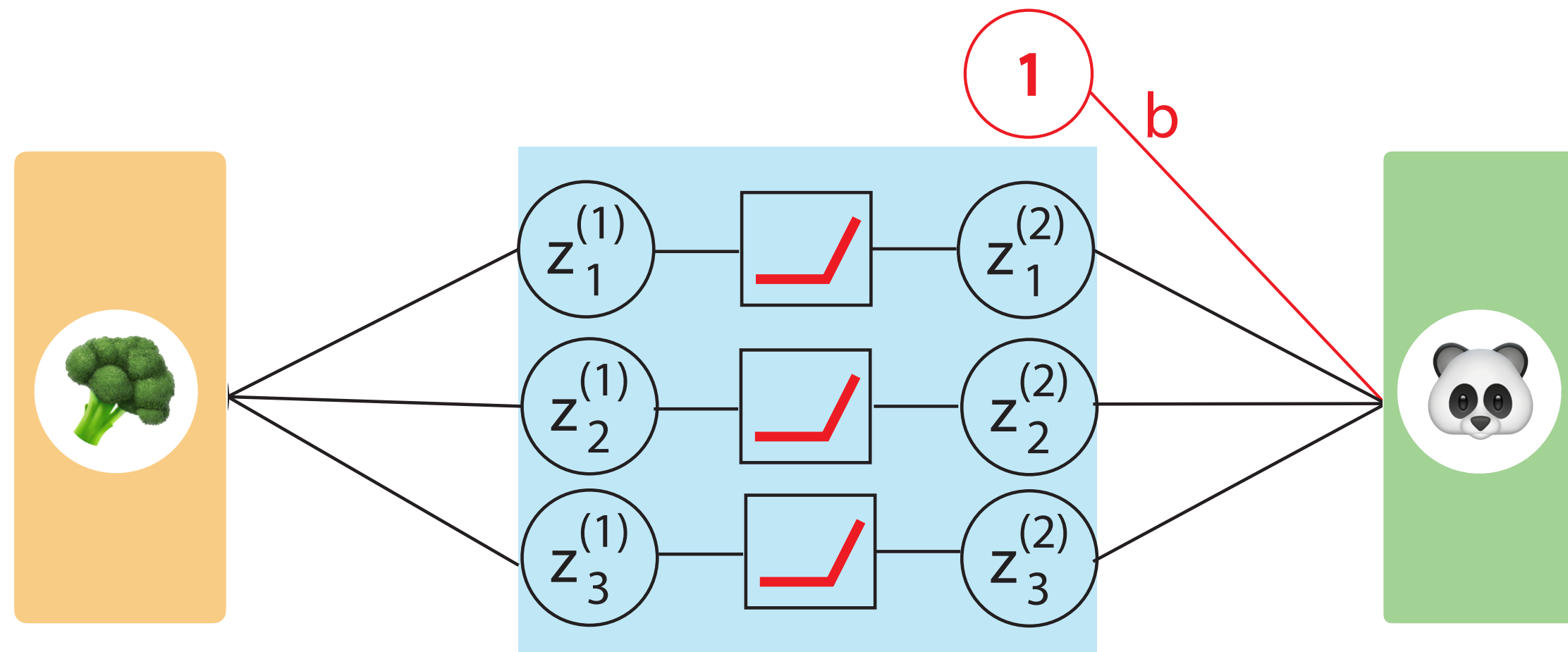
$$P(\text{🐼} | x, w, b, \sigma) \sim \mathcal{N}(y, \sigma)$$



Probability density function of a normal distribution

$$P(\text{🐼} | \text{y}, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{🐼} - \text{y})^2}{2\sigma^2}}$$

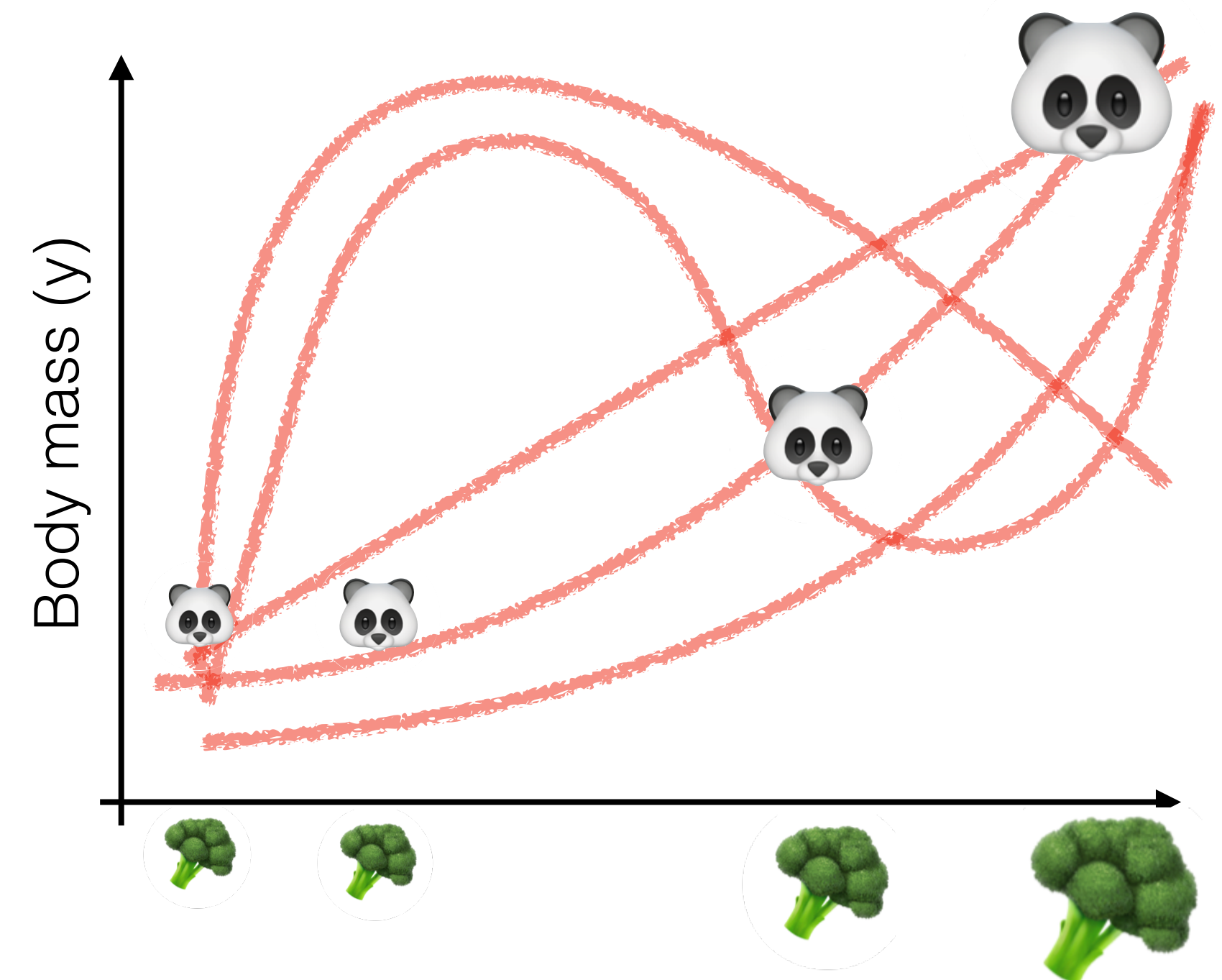
NN regression



Data ('training set')



Optimizing ('training') a model



Likelihood function: the normal density function

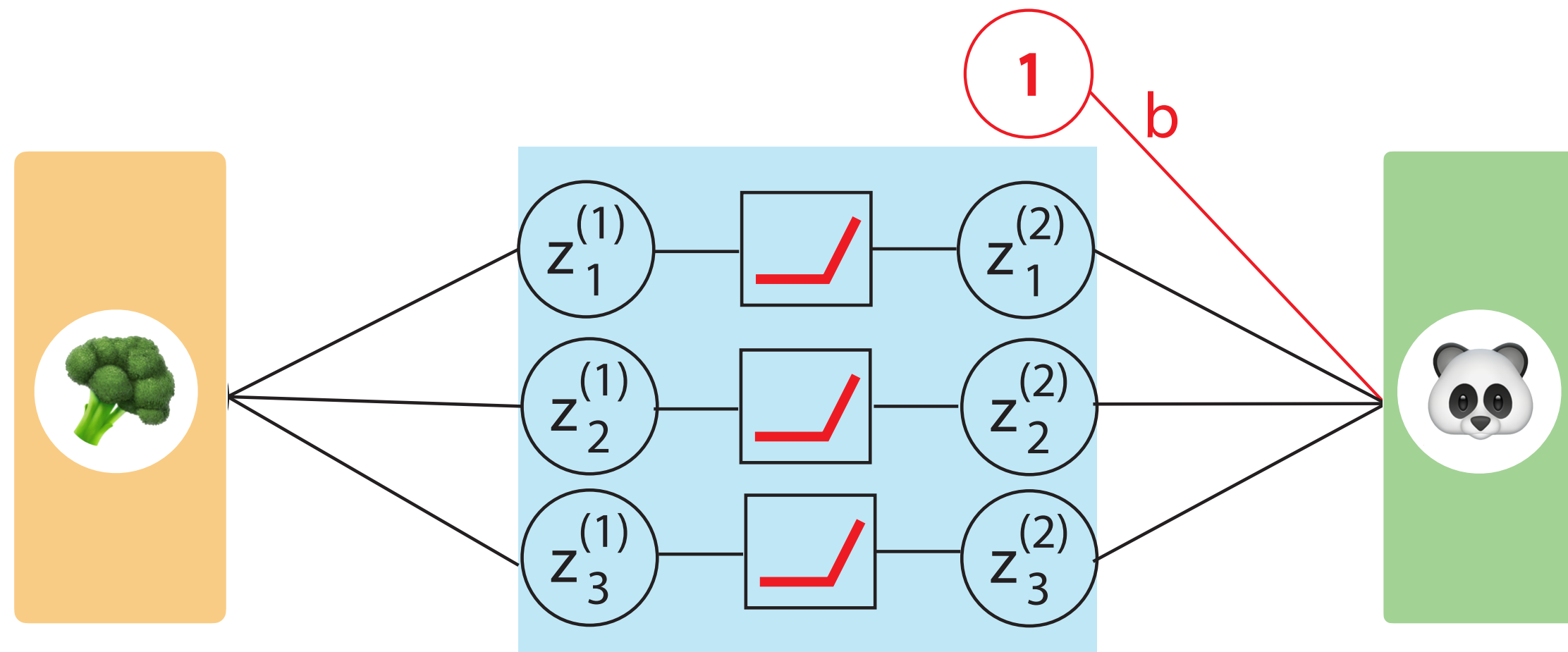
$$P(\text{panda} | \text{broccoli}, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{panda} - \text{broccoli})^2}{2\sigma^2}}$$

Loss function: mean squared error

$$-\log P(\text{panda} | \text{broccoli}) \propto (\text{panda} - \text{broccoli})^2$$

Squared difference between truth and prediction

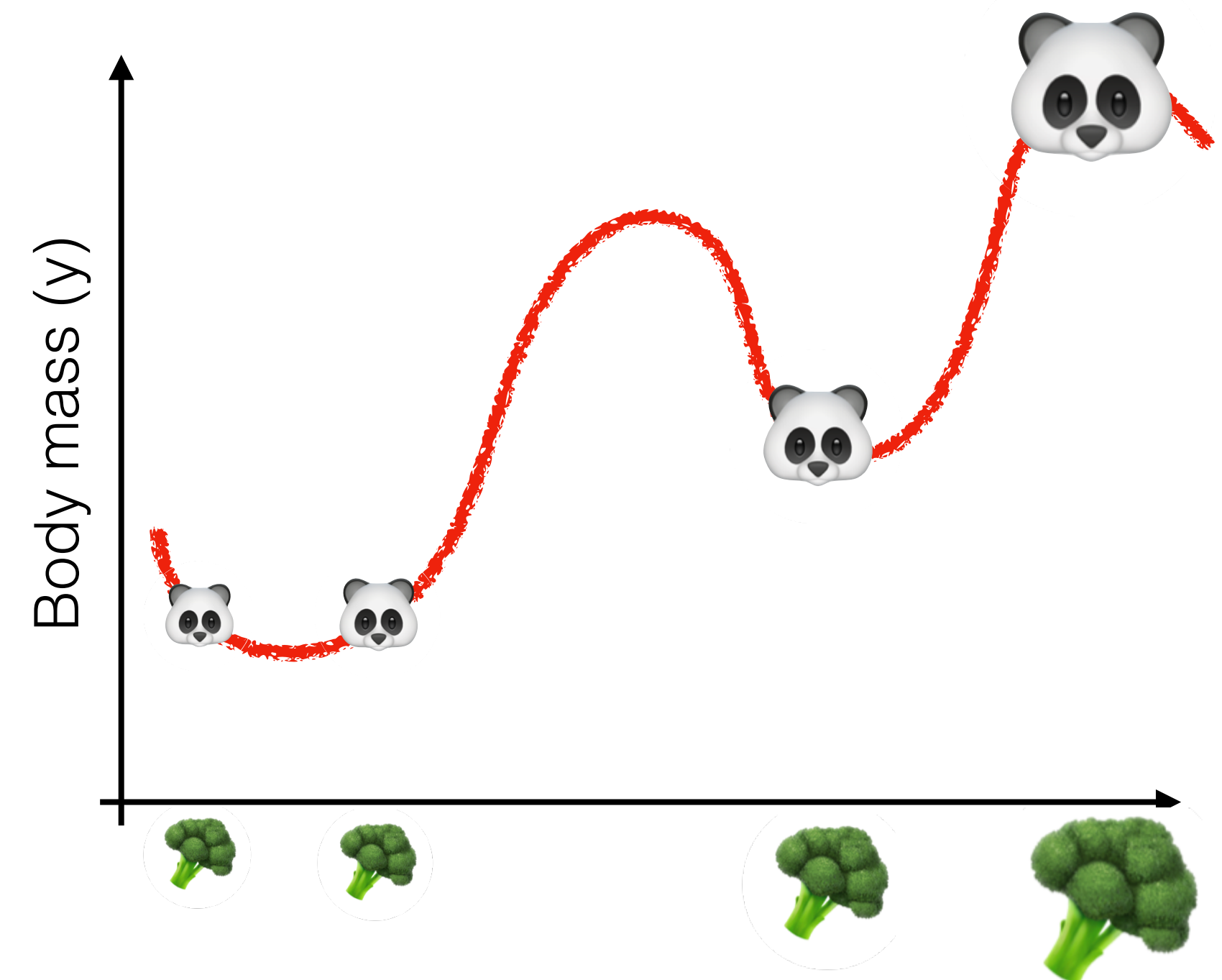
NN regression



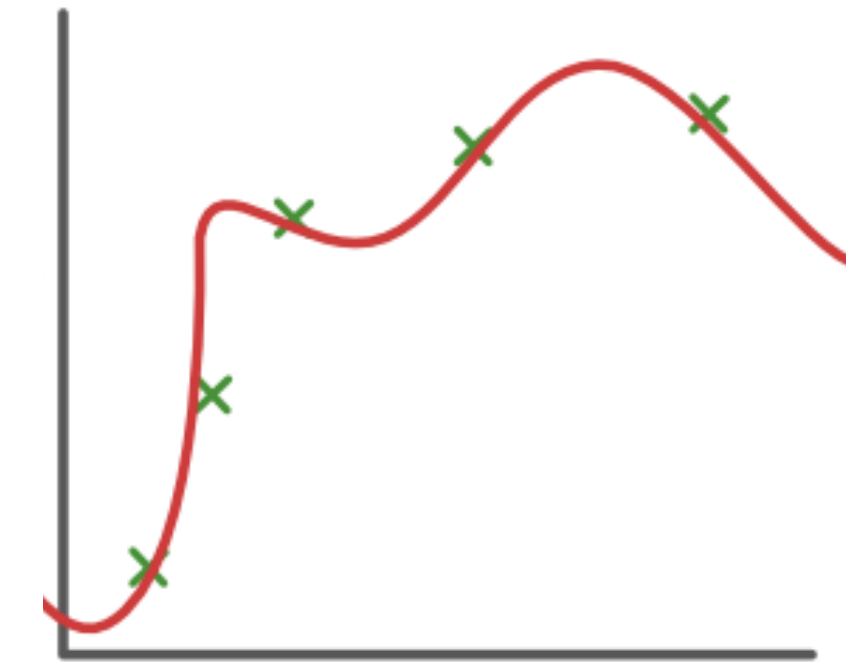
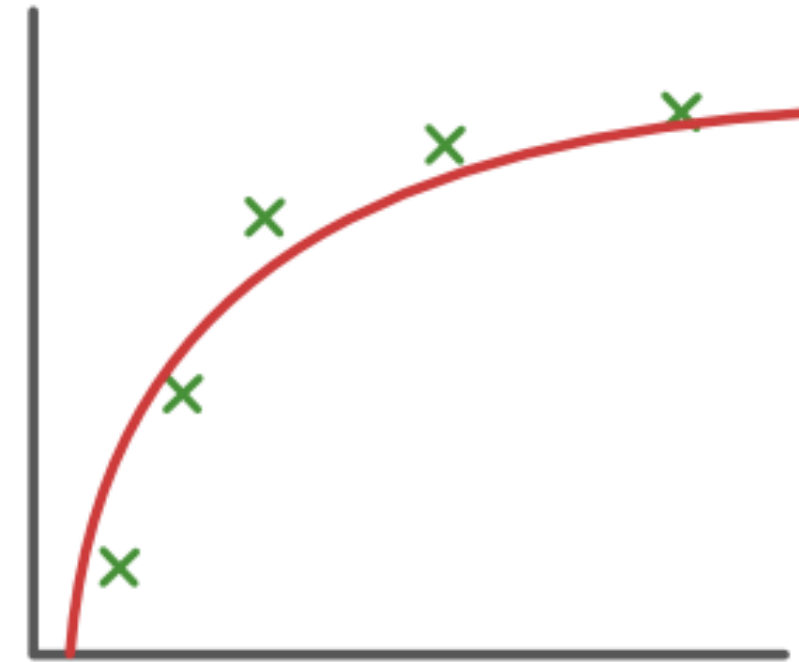
Data ('training set')



Best-fitting model



Neural networks are over-parameterized models and will overfit with a maximum likelihood approach



Statistical inference:

Step 1: select best model while penalizing for complexity (e.g. AIC or Bayes factors)

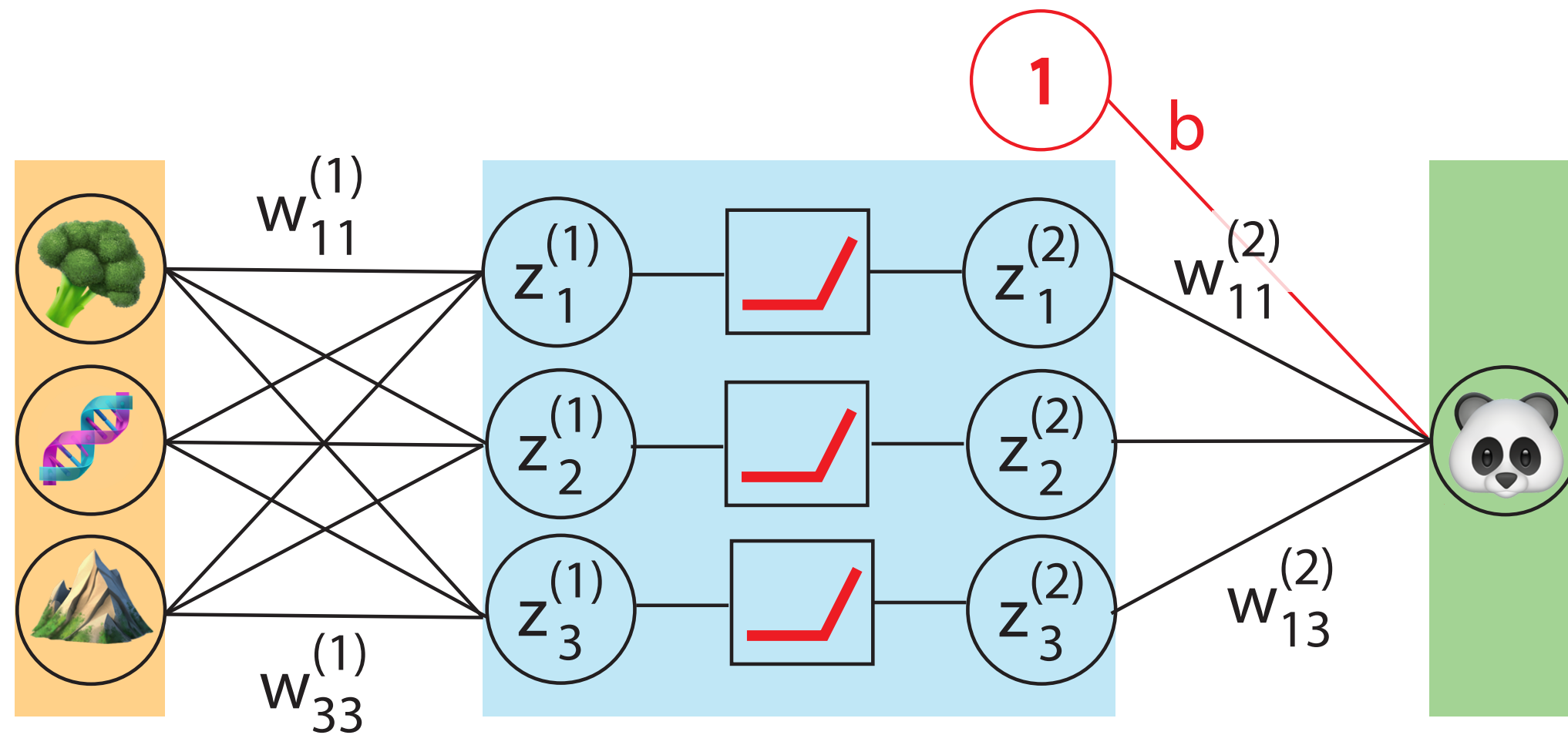
Step 2: find the parameters that maximize the likelihood of your data under the model

ML inference:

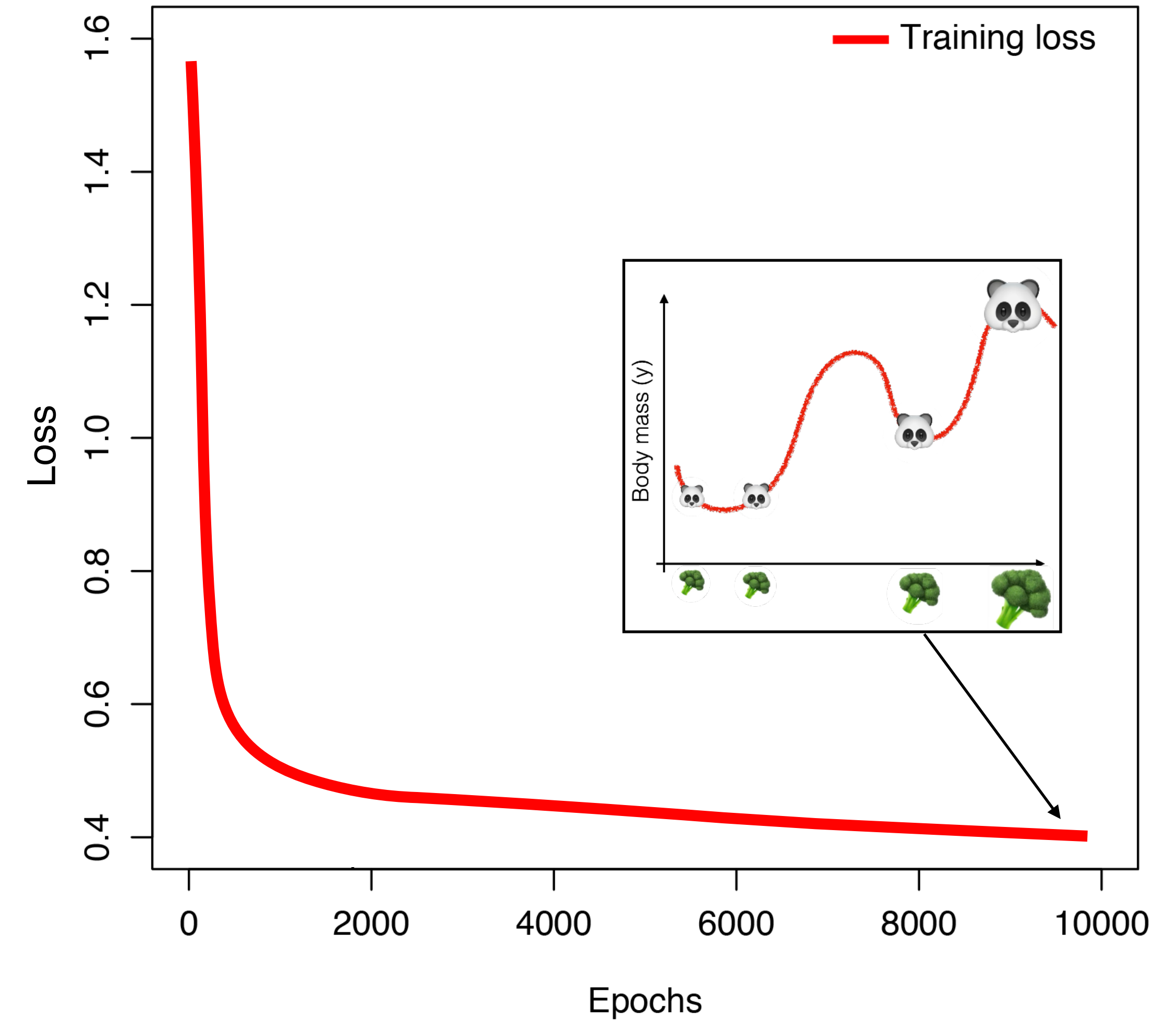
Step 1: fit your model while preventing overfitting (i.e. stop optimizing before reaching the maximum likelihood)

Step 2 (optional): find the hyper-parameters that give you the best results (e.g. based on a test set)

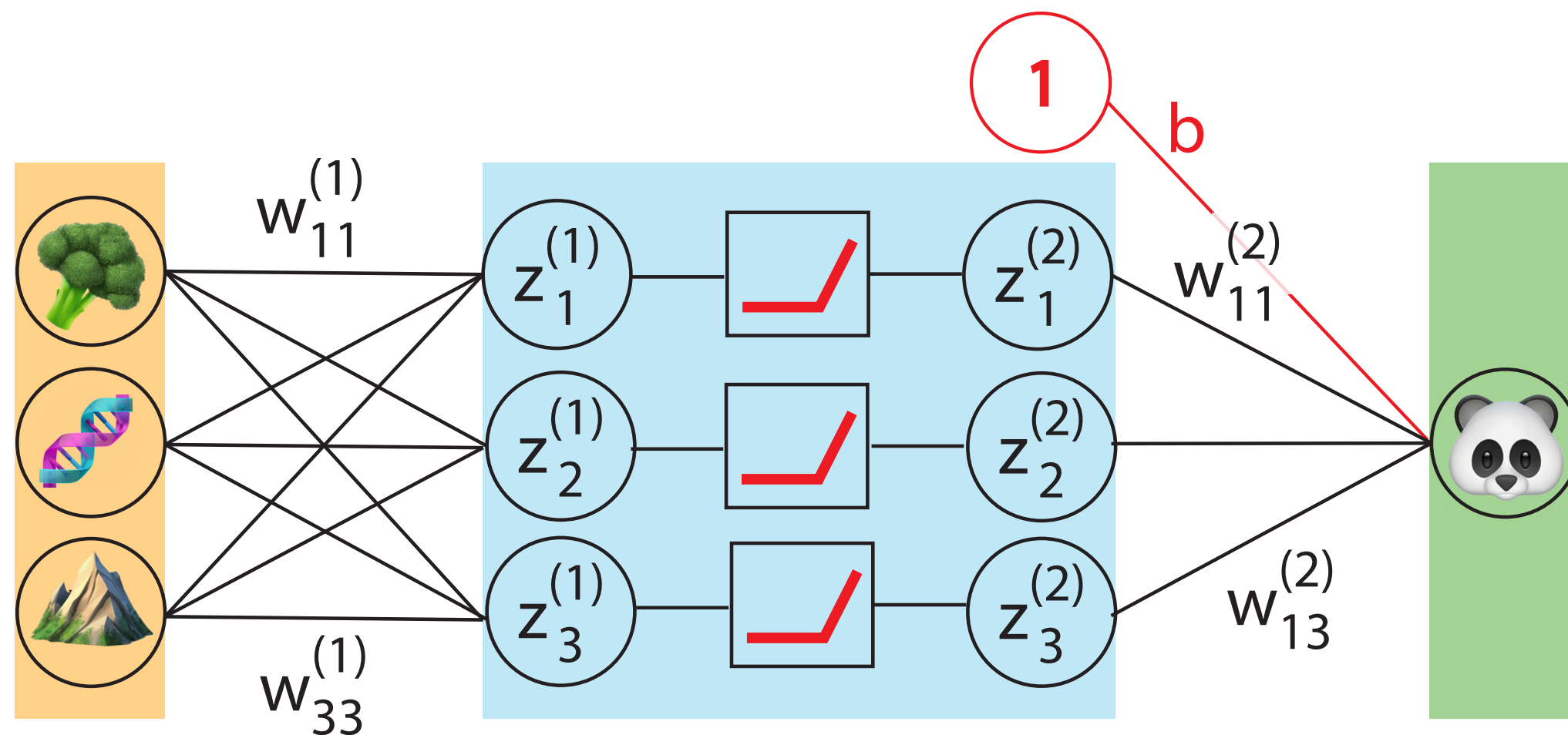
Optimization of a NN model



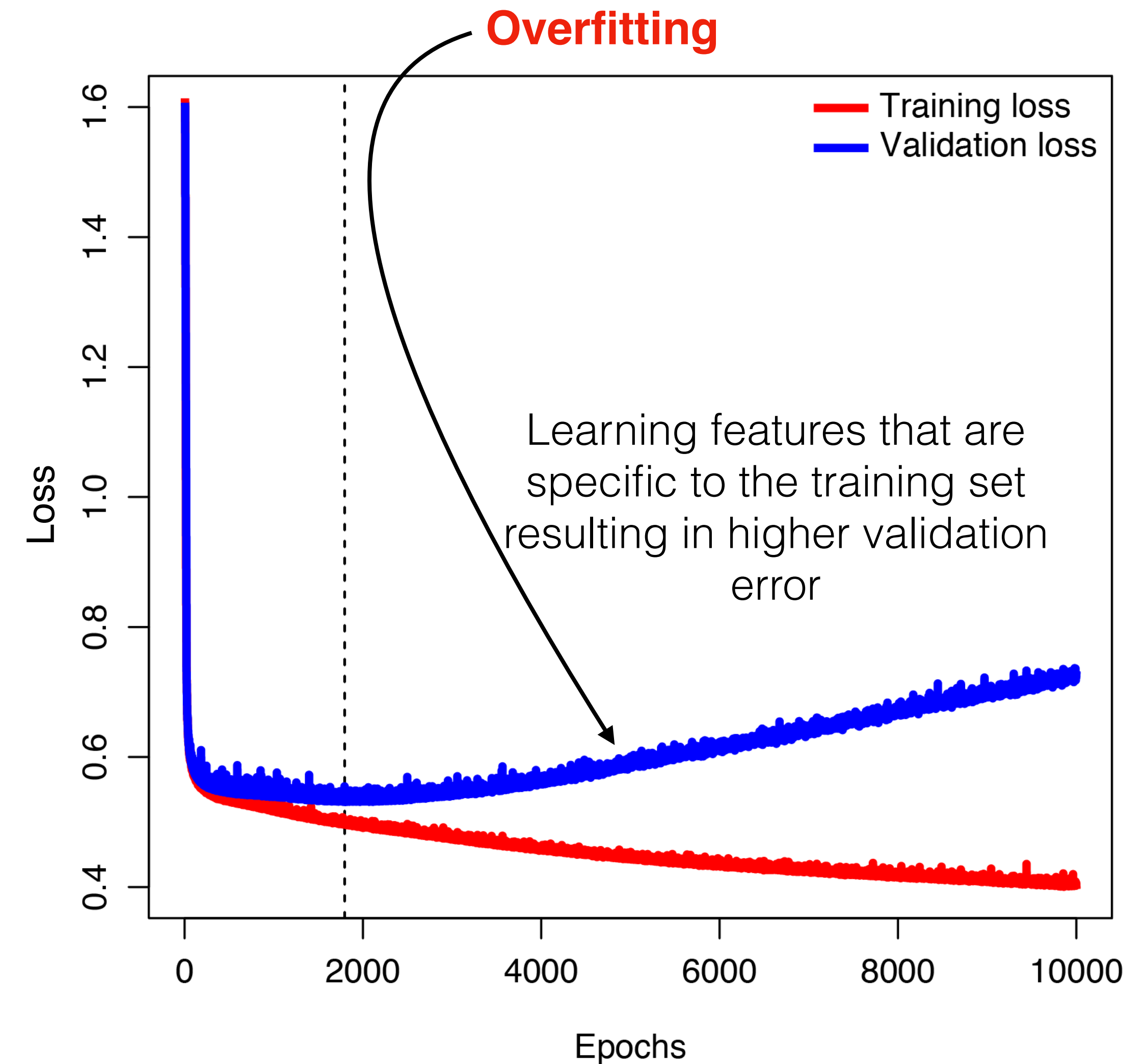
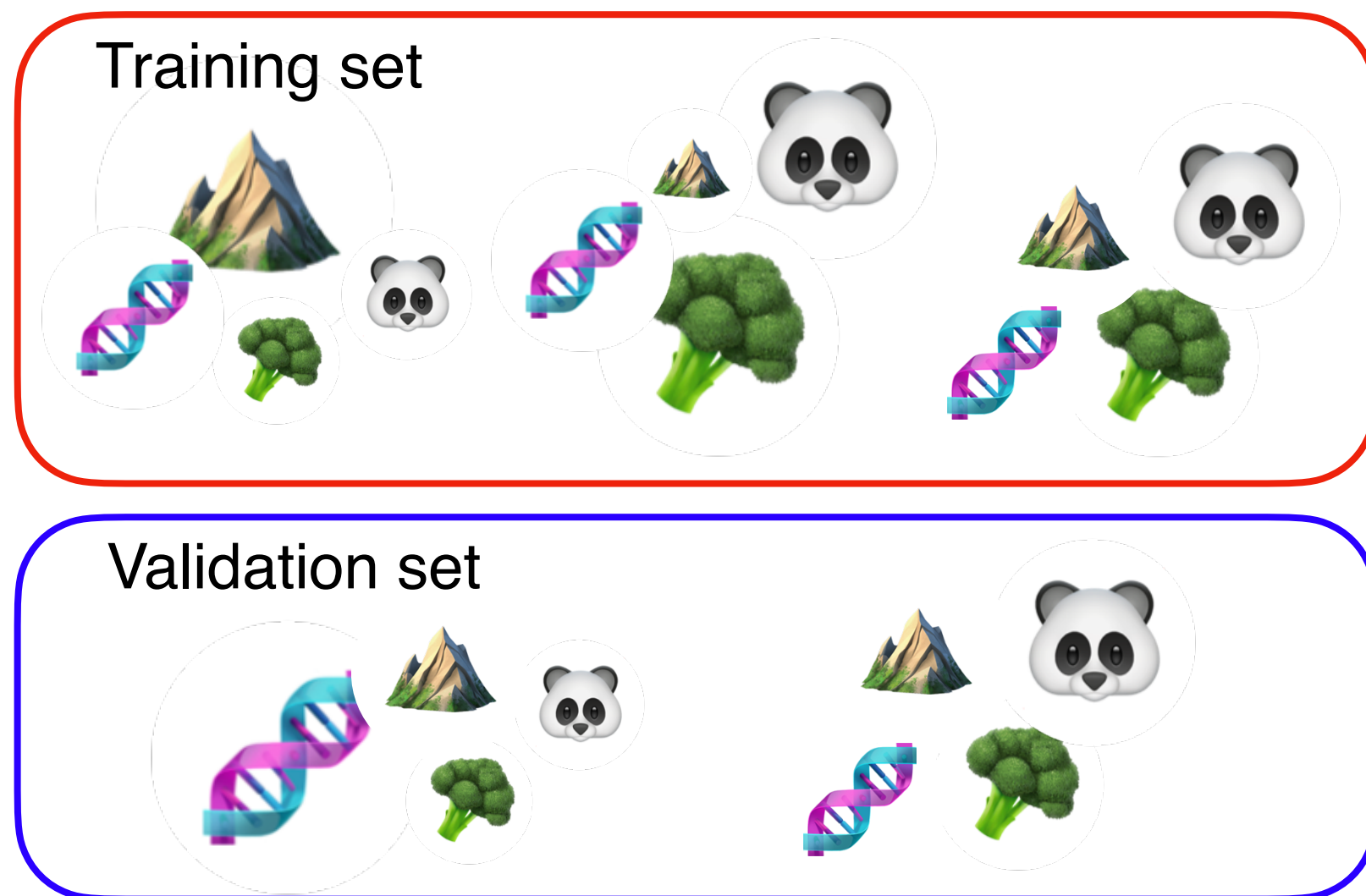
A maximum likelihood (or minimum loss) optimization of an NN would inevitably lead to over-fitting because NNs are over-parameterized



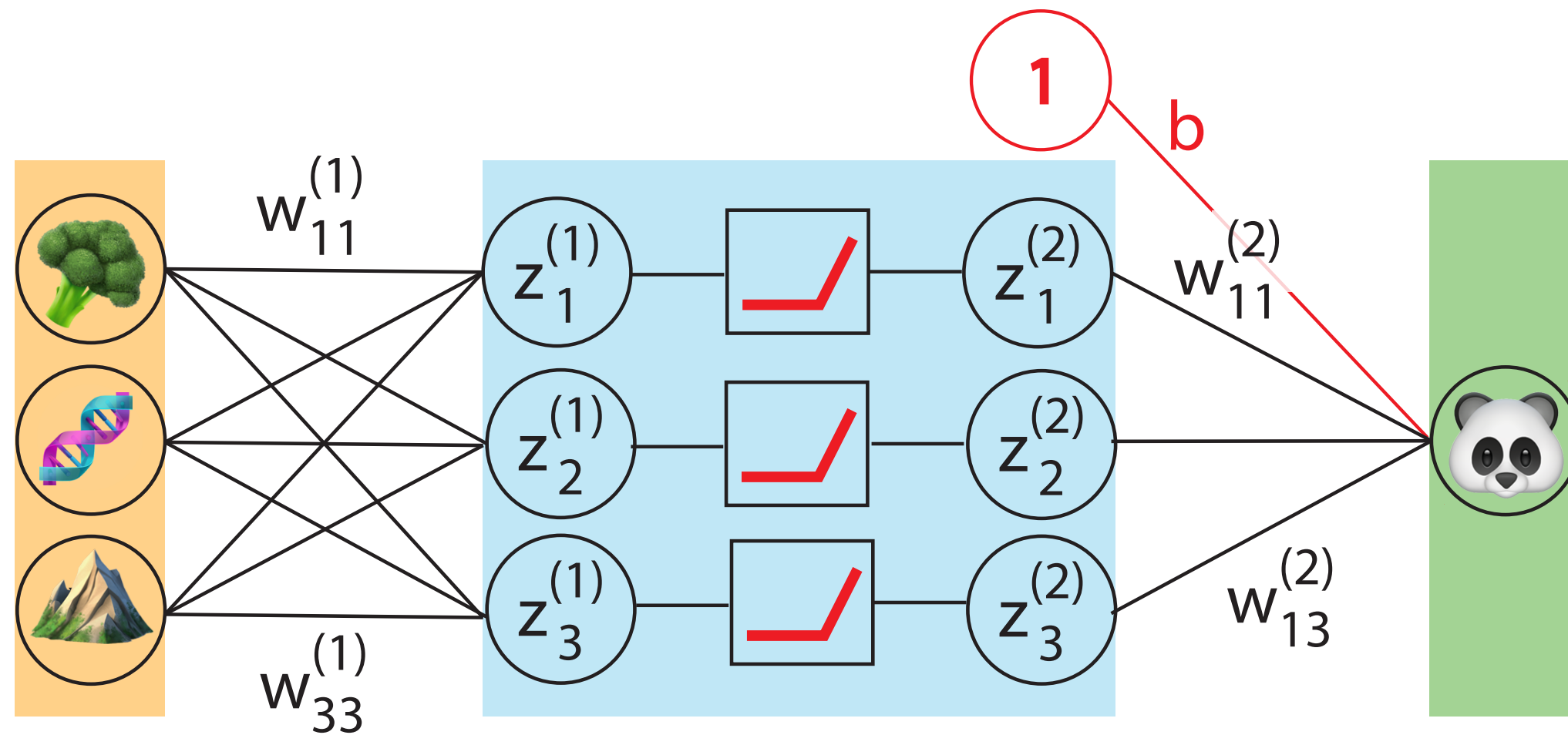
Optimization of a NN model



In machine learning a separate validation set is used to stop the optimization process before it starts overfitting



Optimization of a NN model

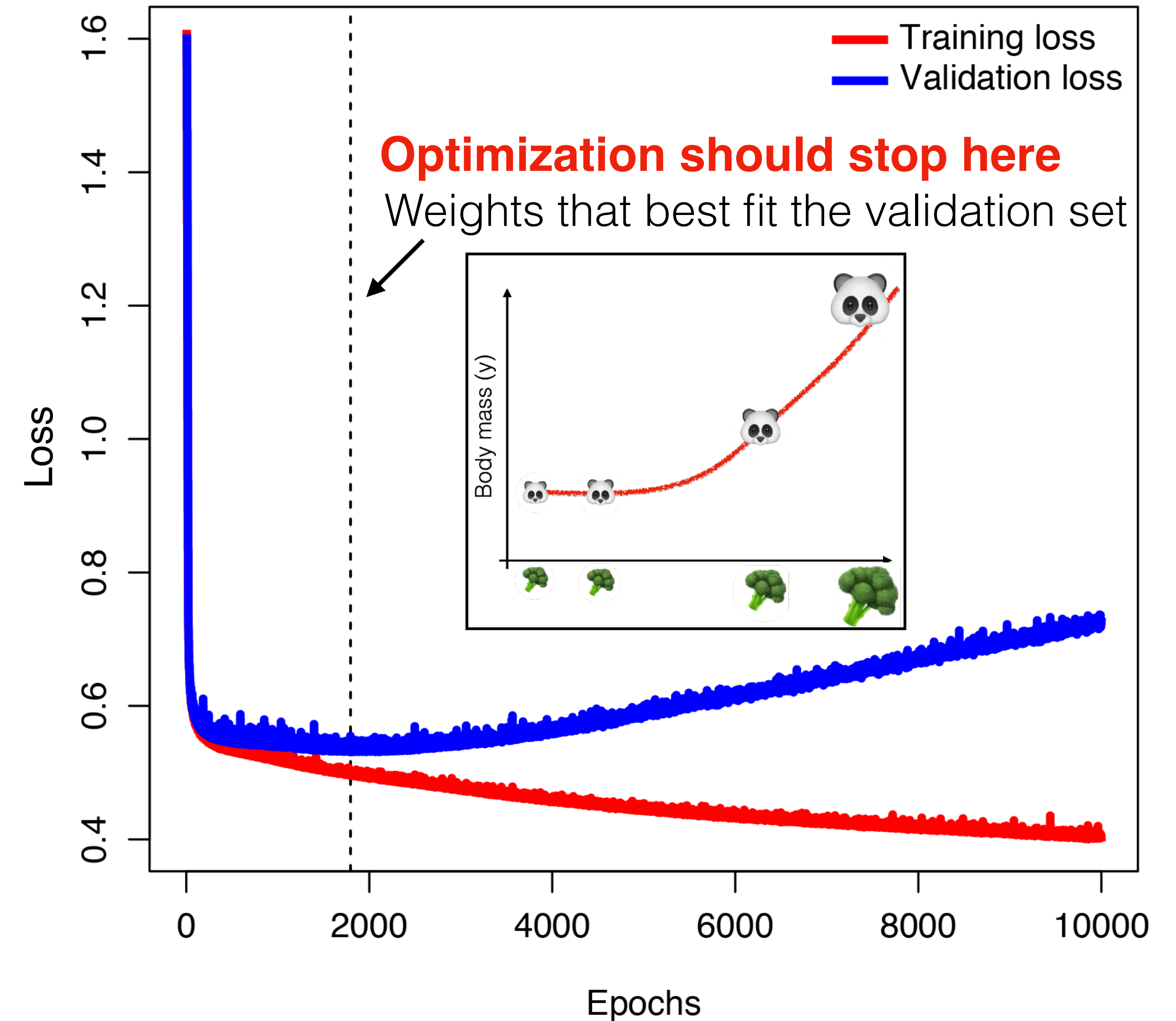


In machine learning a separate validation set is used to stop the optimization process before it starts overfitting

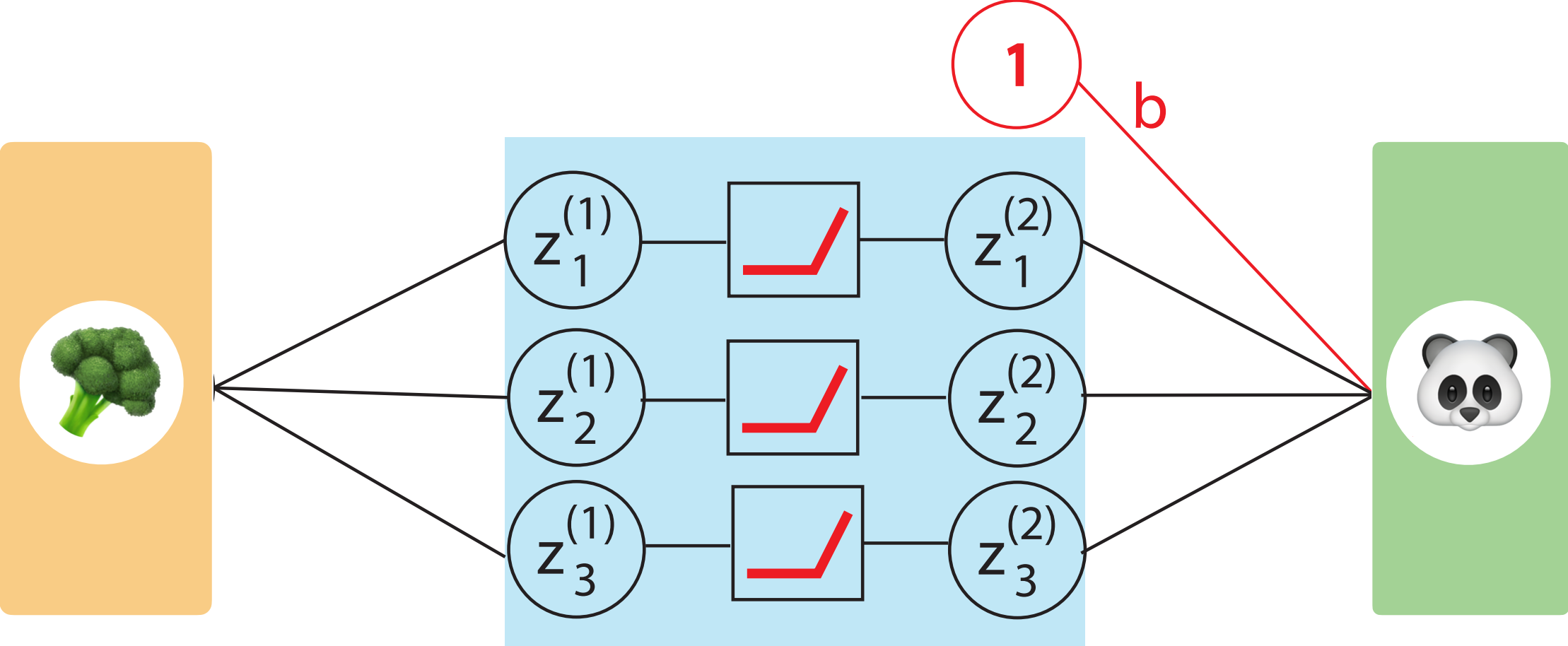
Training set



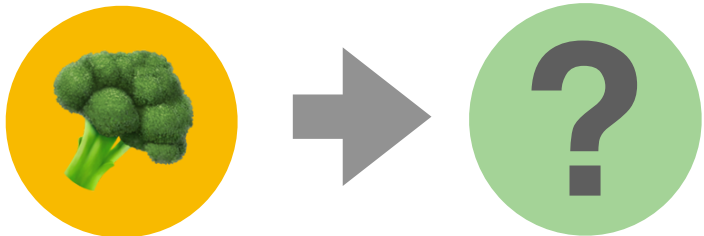
Validation set



NN regression



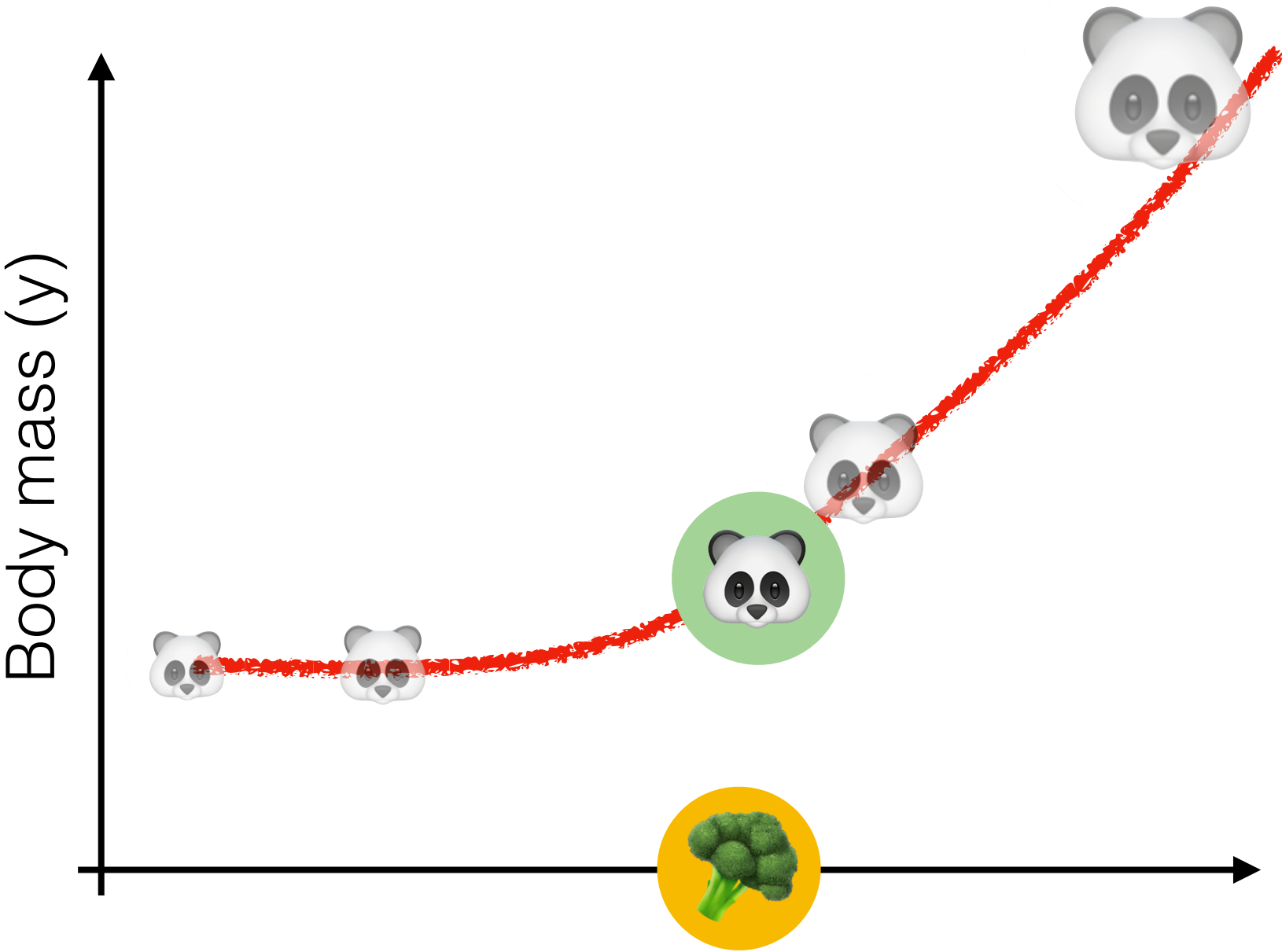
Predicting body mass from diet



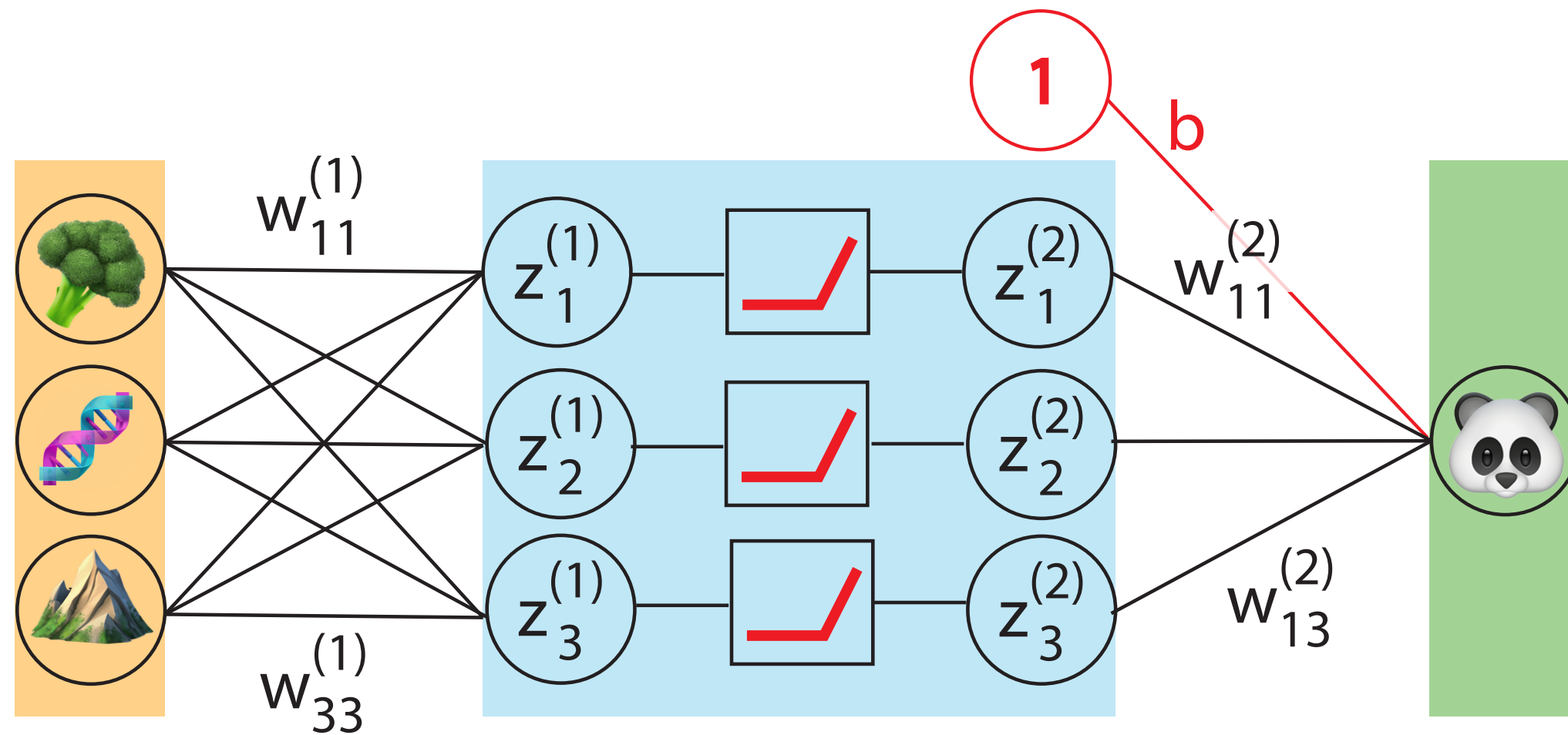
Data ('training set')



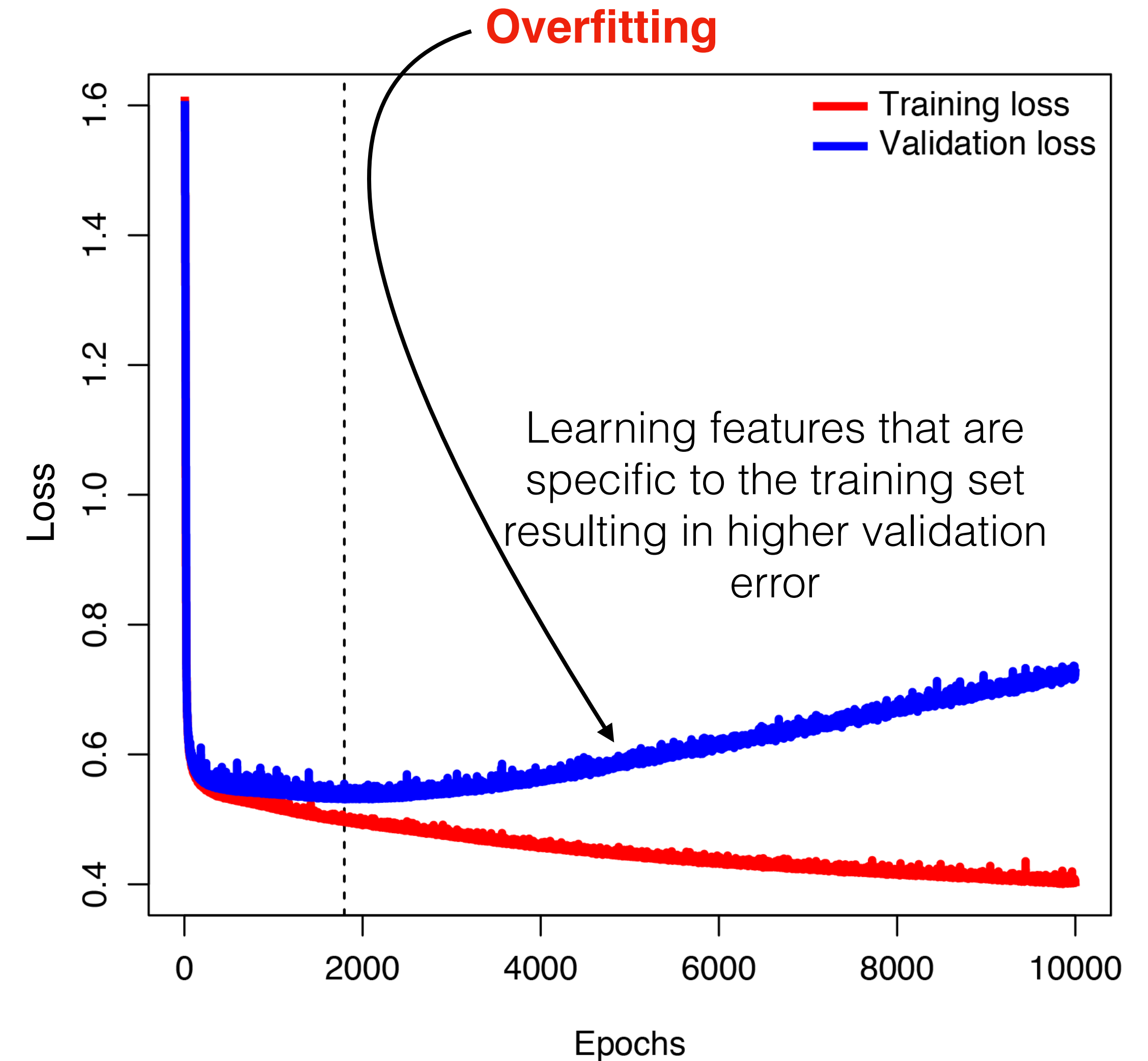
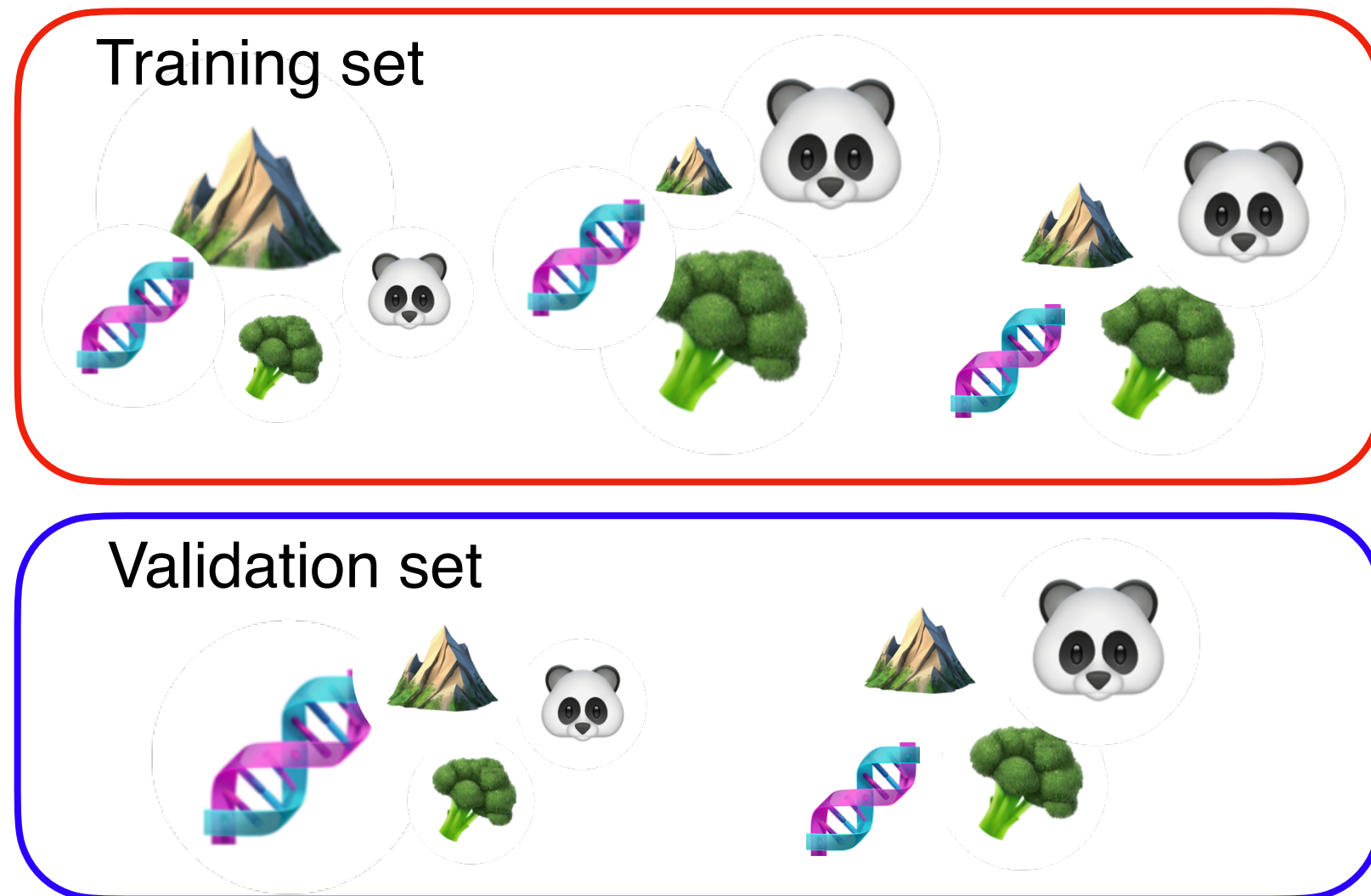
Trained model



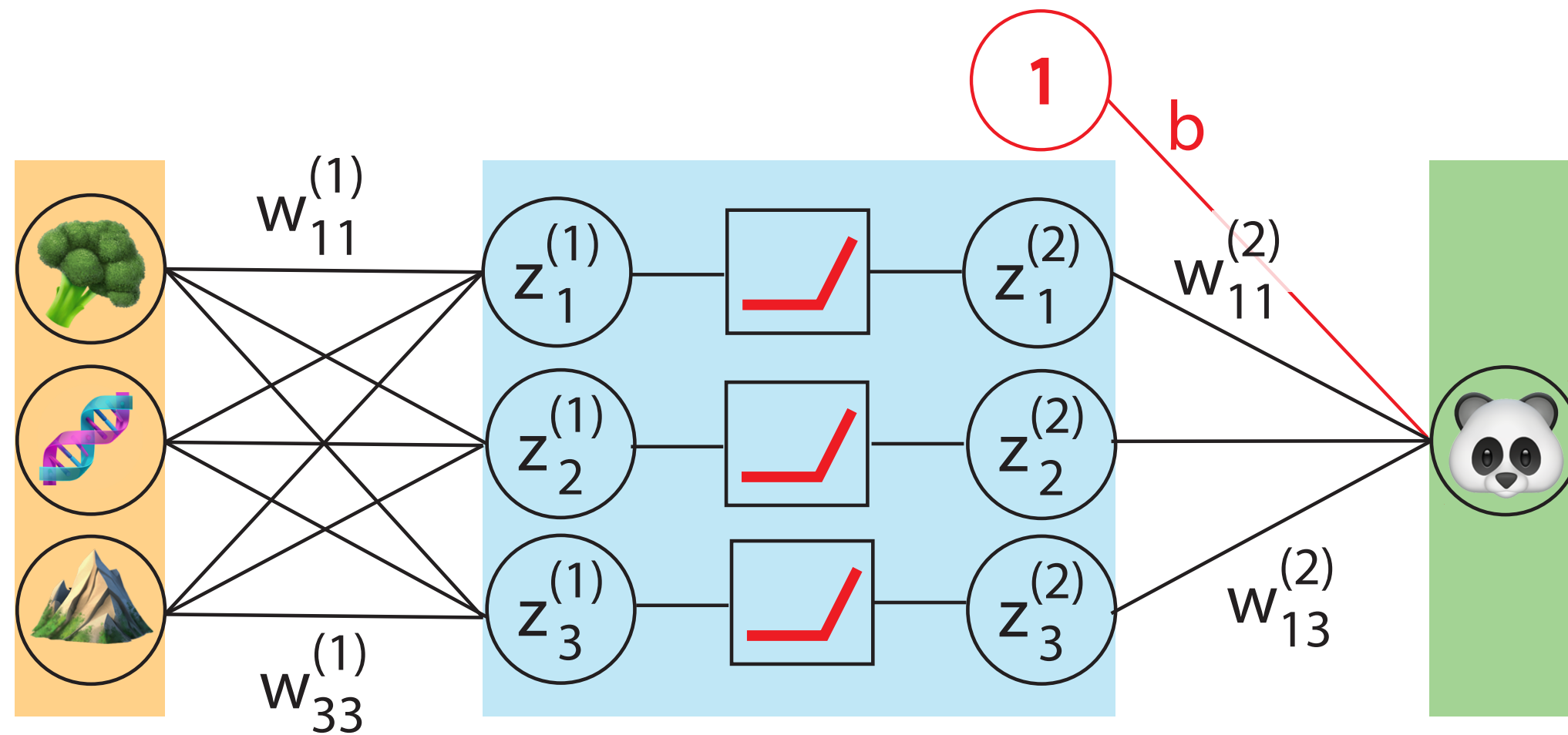
Optimization of a NN model



In machine learning a separate validation set is used to stop the optimization process before it starts overfitting

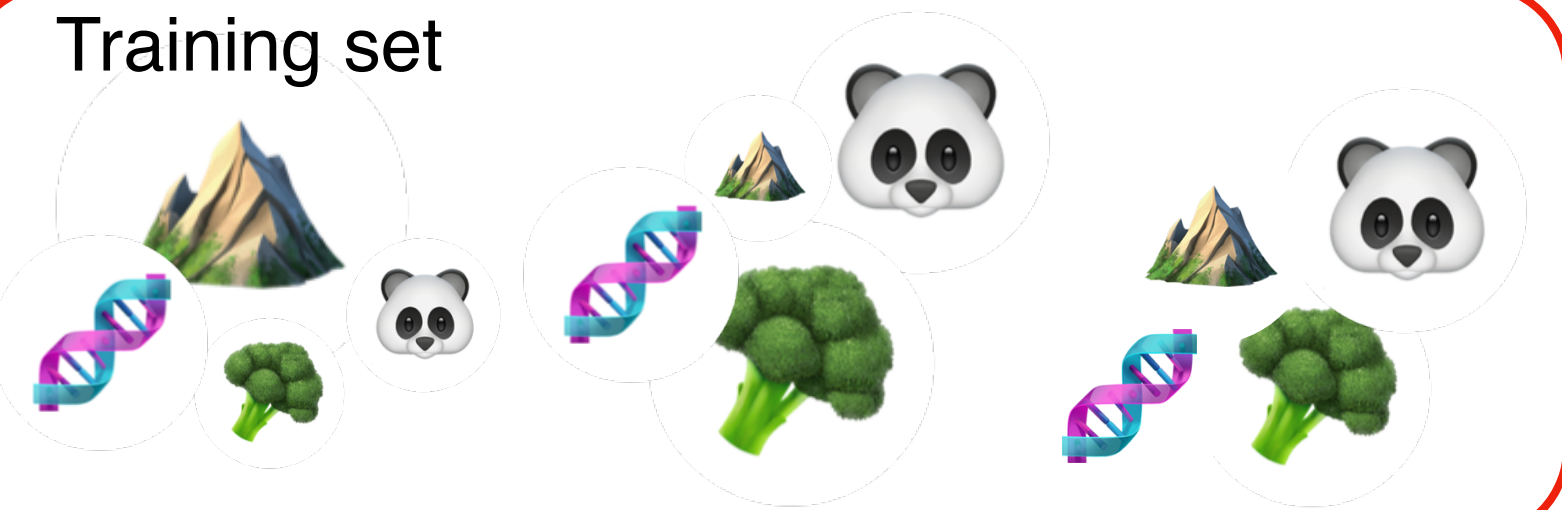


Optimization of a NN model

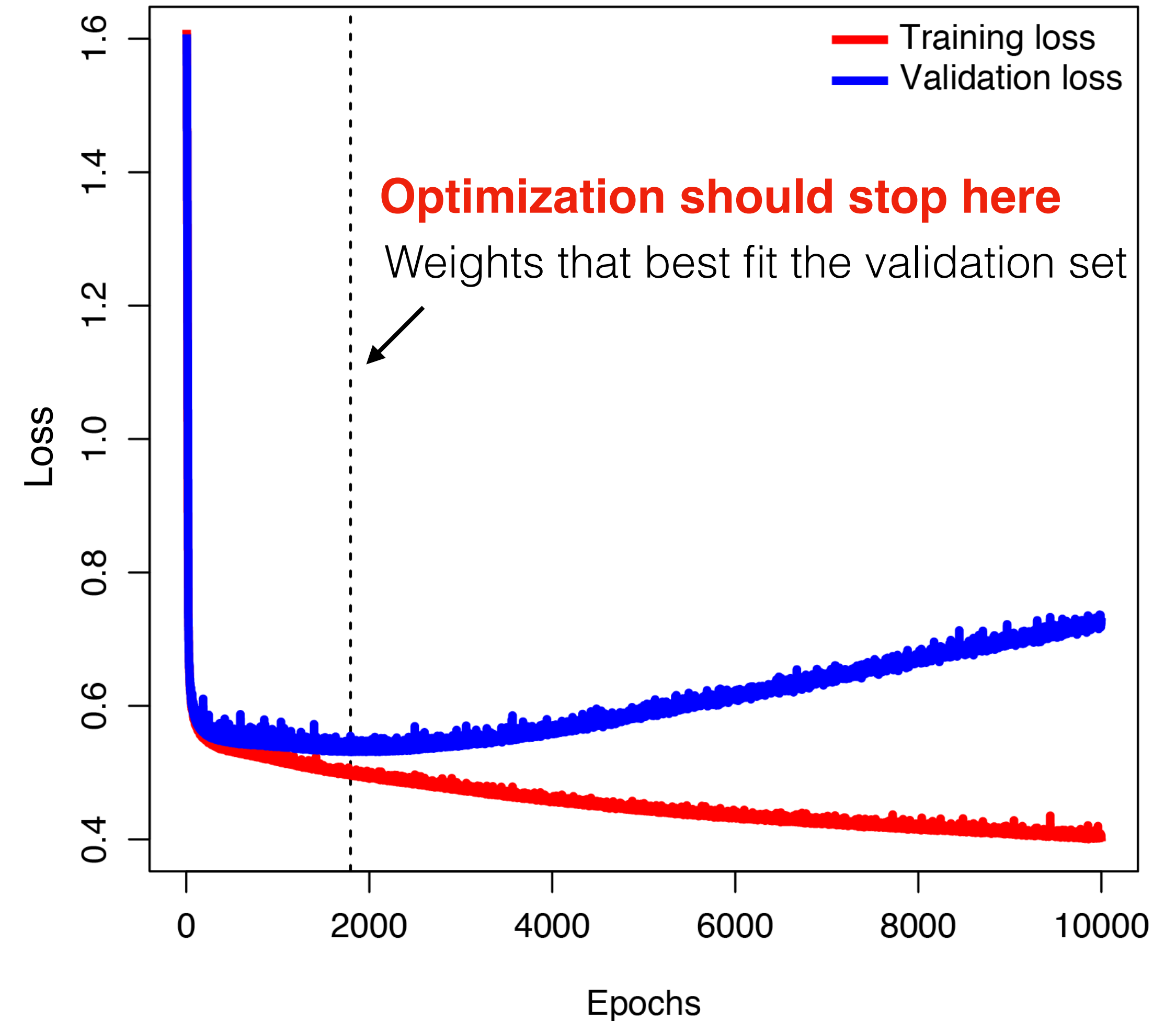
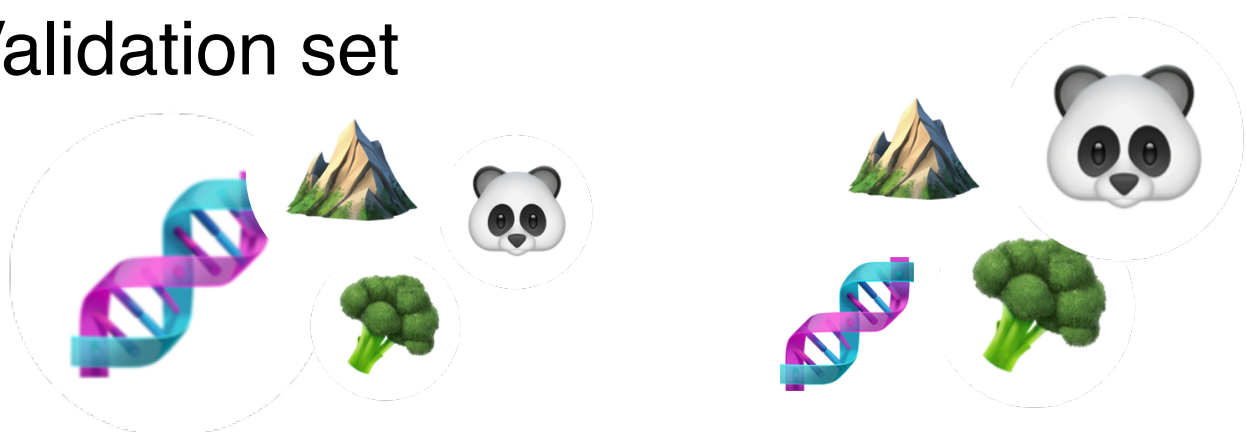


In machine learning a separate validation set is used to stop the optimization process before it starts overfitting

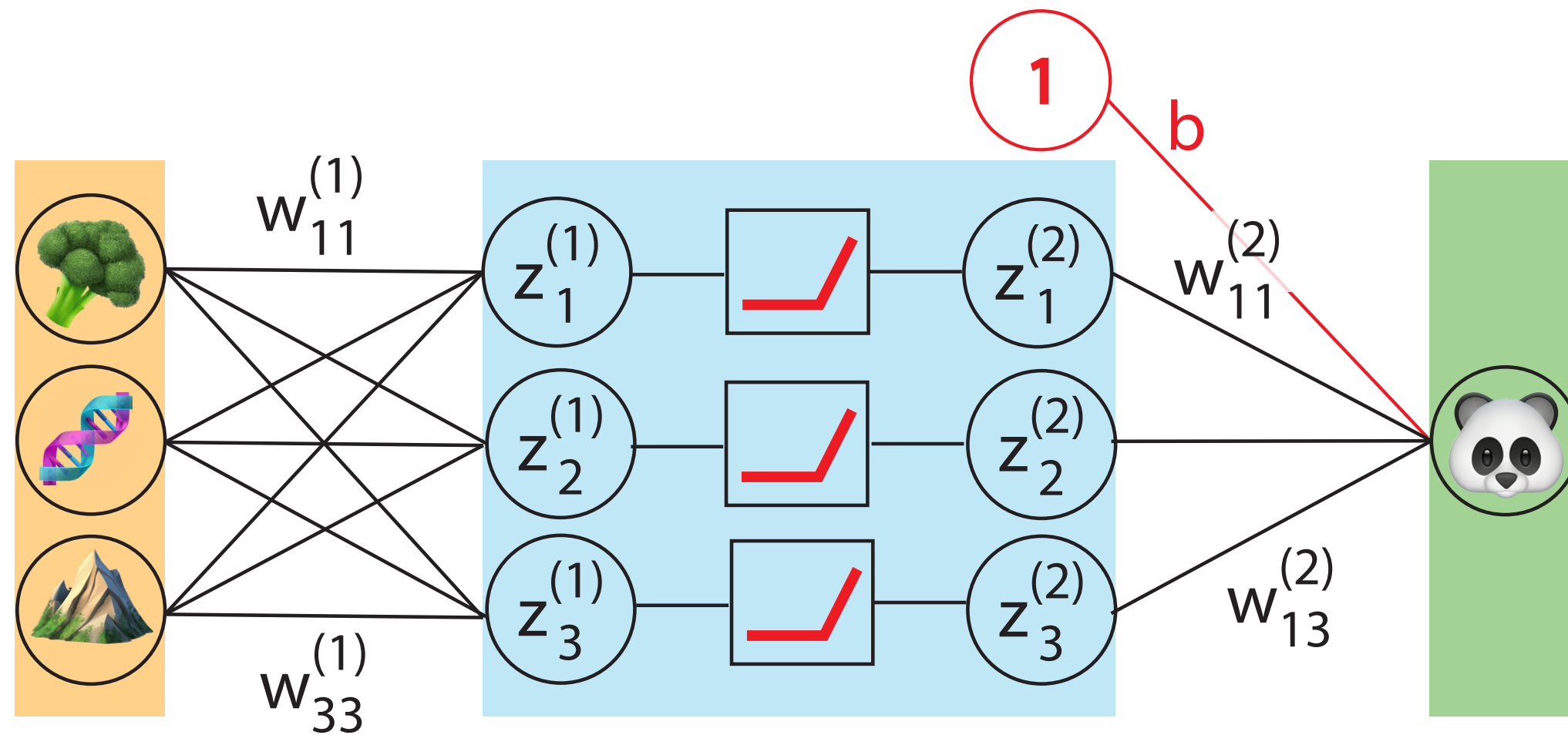
Training set



Validation set



Likelihood in a neural network model of classification



In a BNN the priors have a regularizing effect that limits the risk of overfitting

Priors on the weights

$$P(w) \sim \mathcal{N}(0, 1)$$

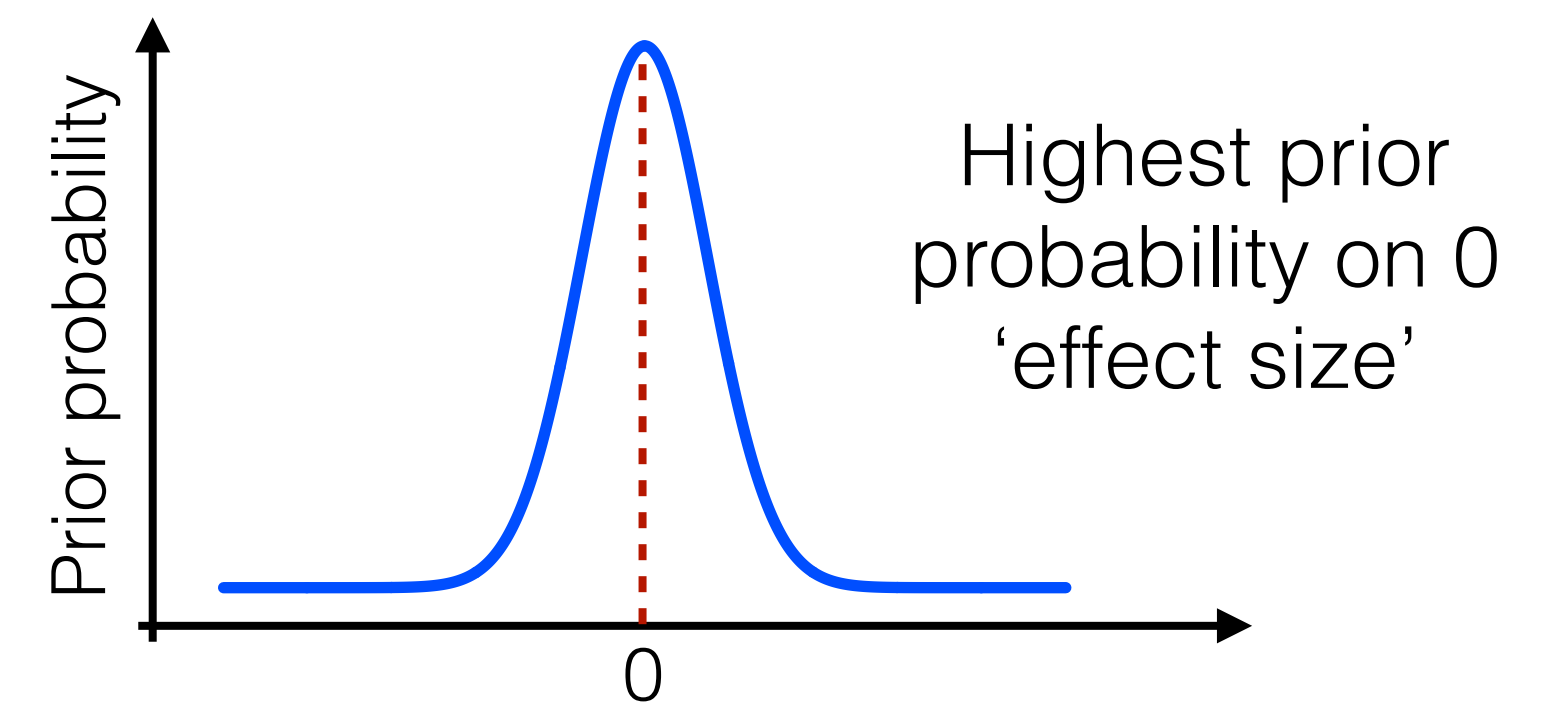
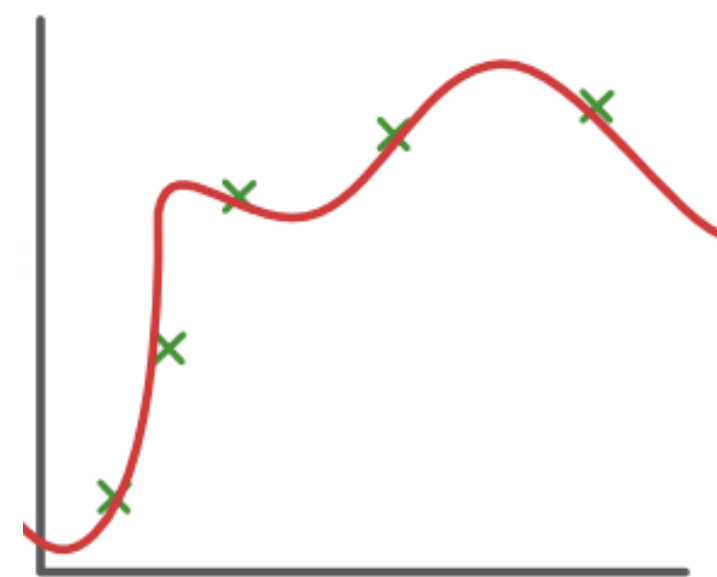
Under-fitting



Just right

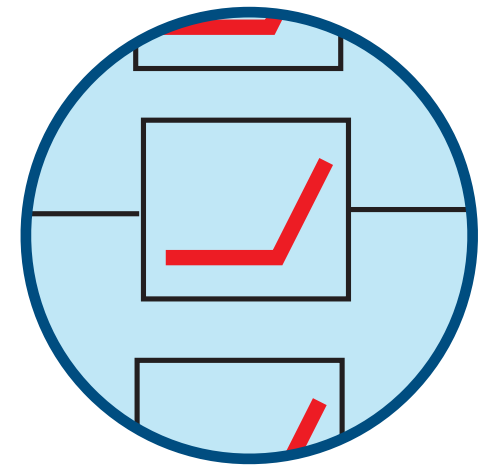


Over-fitting



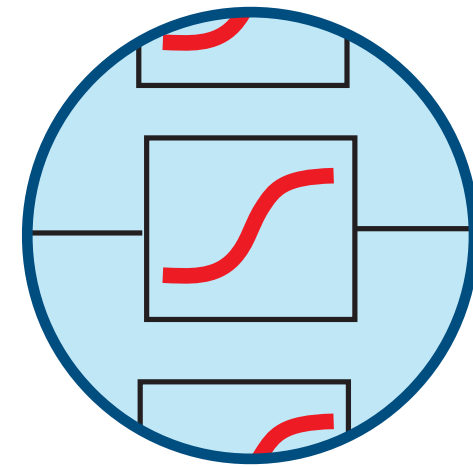
Parameterization of a neural network

Activation functions



ReLU: rectified linear unit

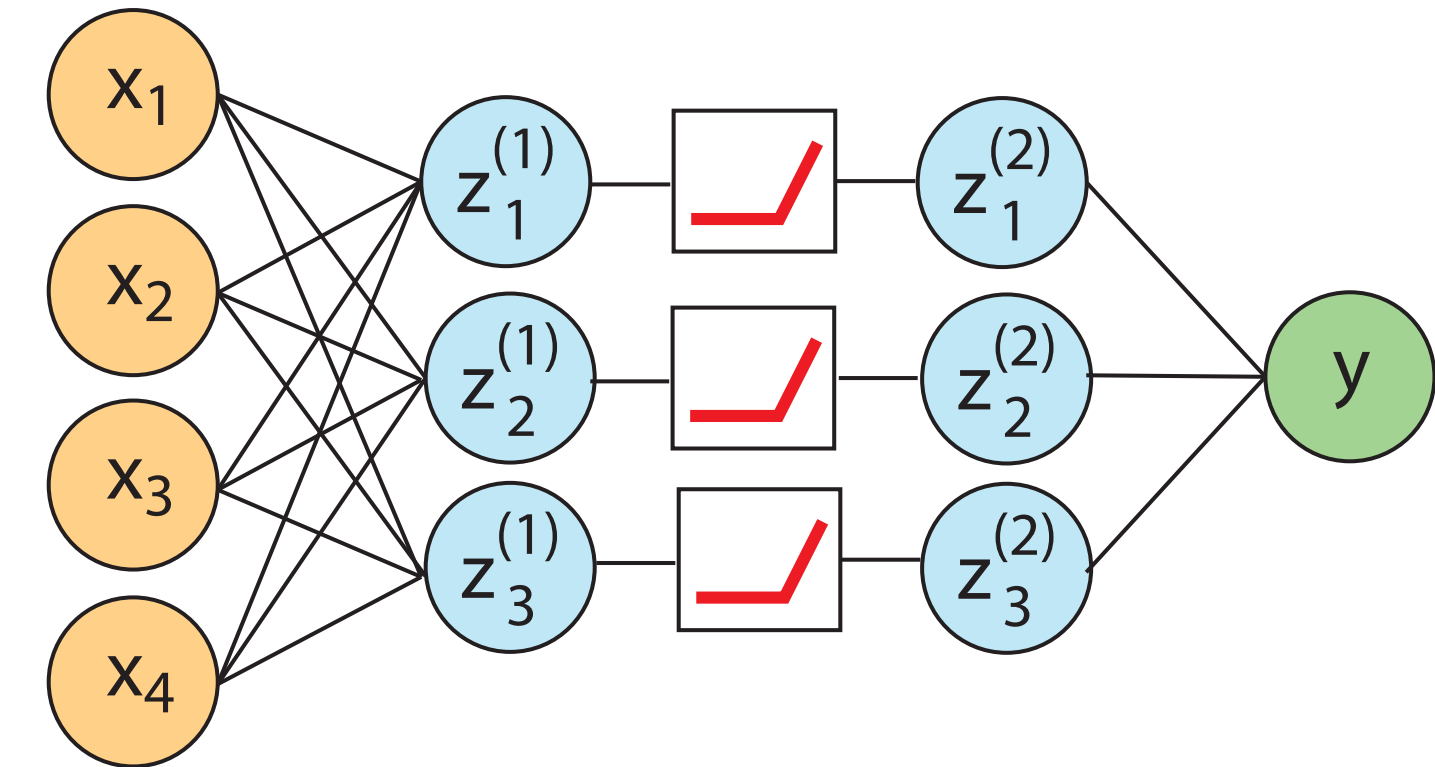
$$\text{ReLU}(x) = \max(0, x)$$



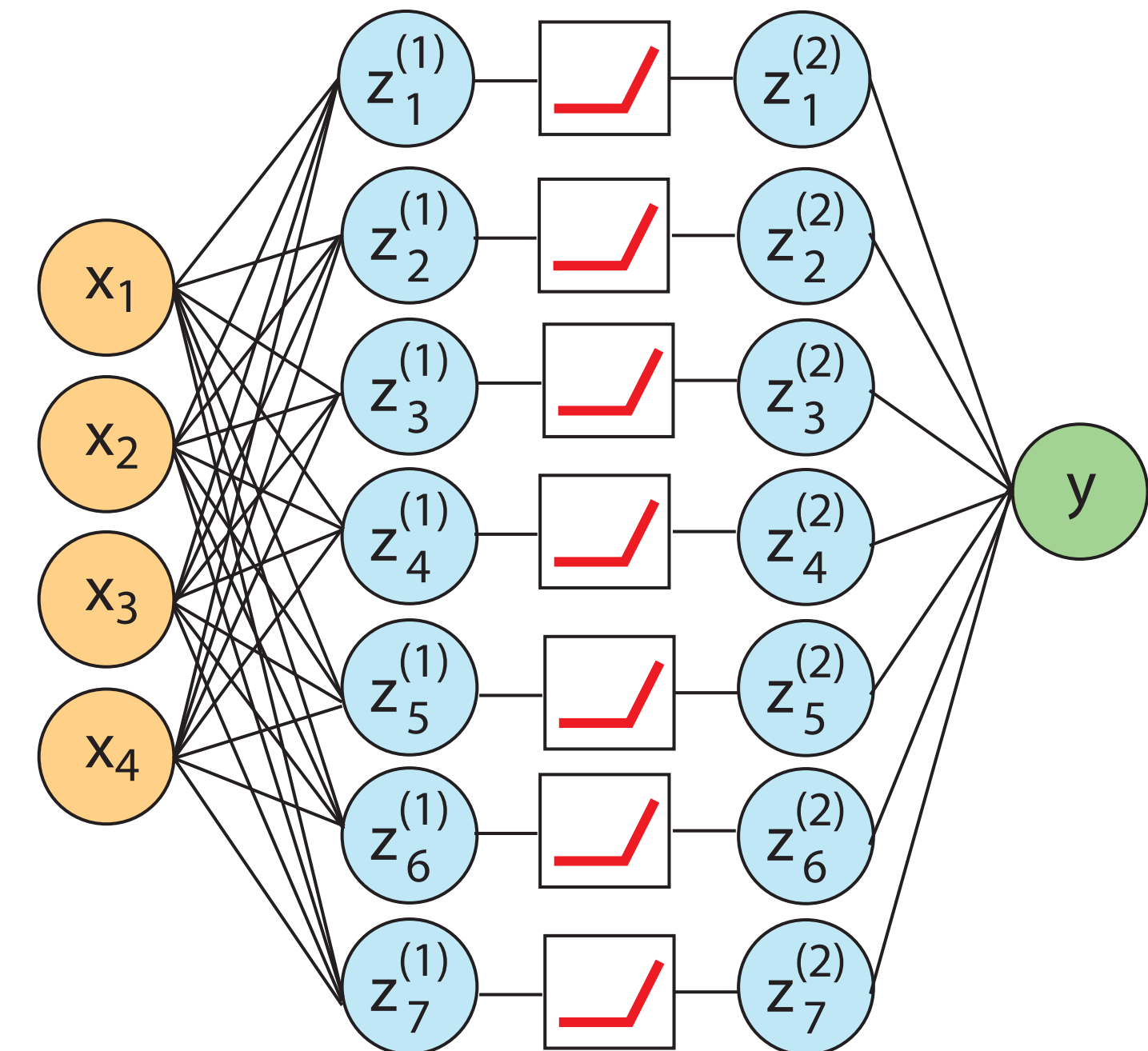
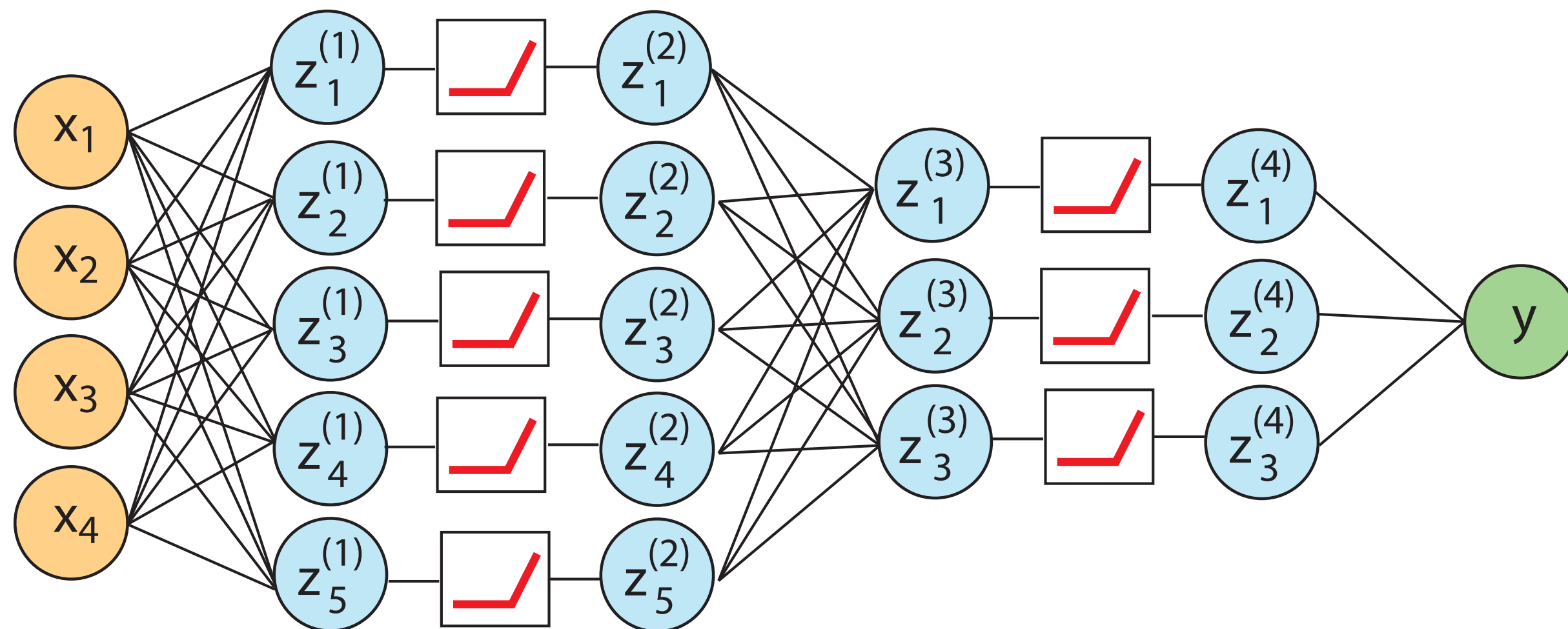
Sigmoid

$$\text{sigmoid}(x) = 1 / (1 + \exp(-x))$$

Number of nodes

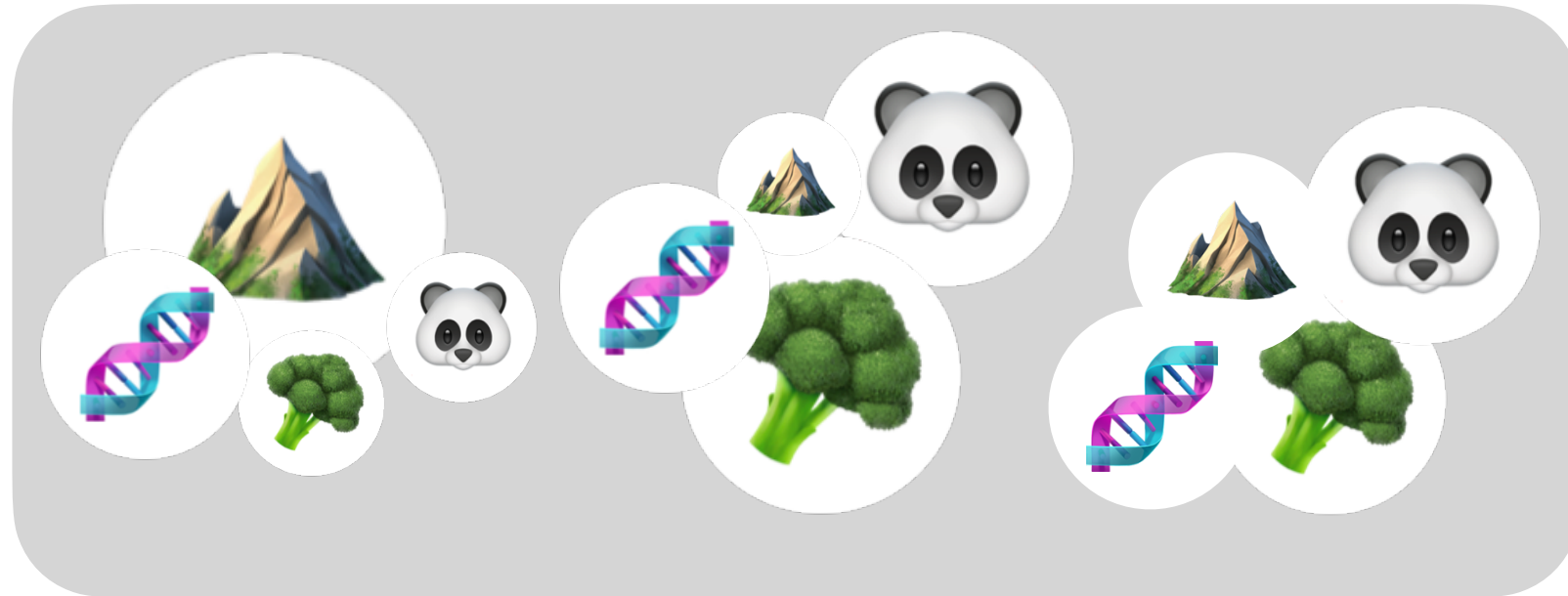


Number of hidden layers (deep NNs)

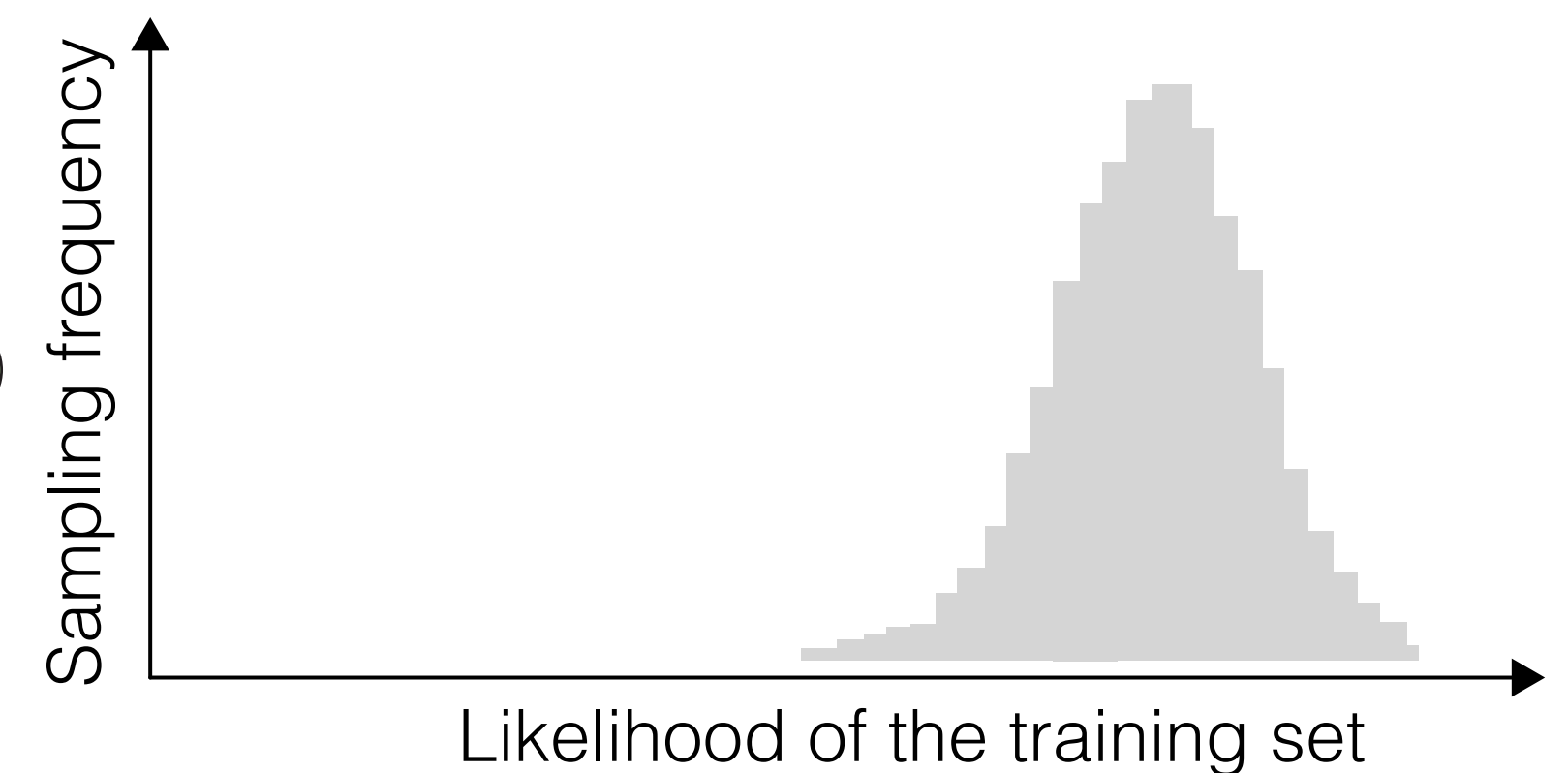
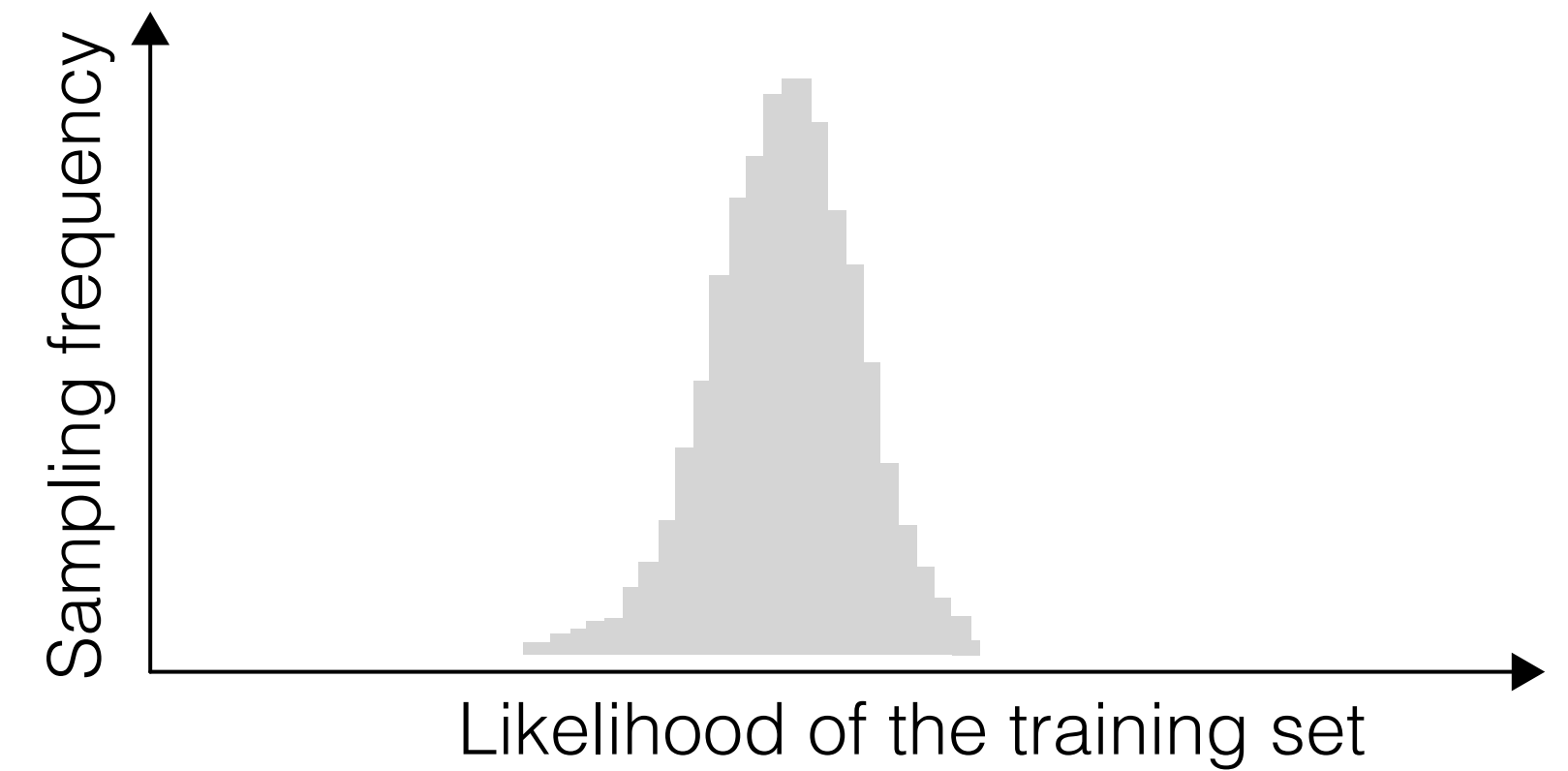
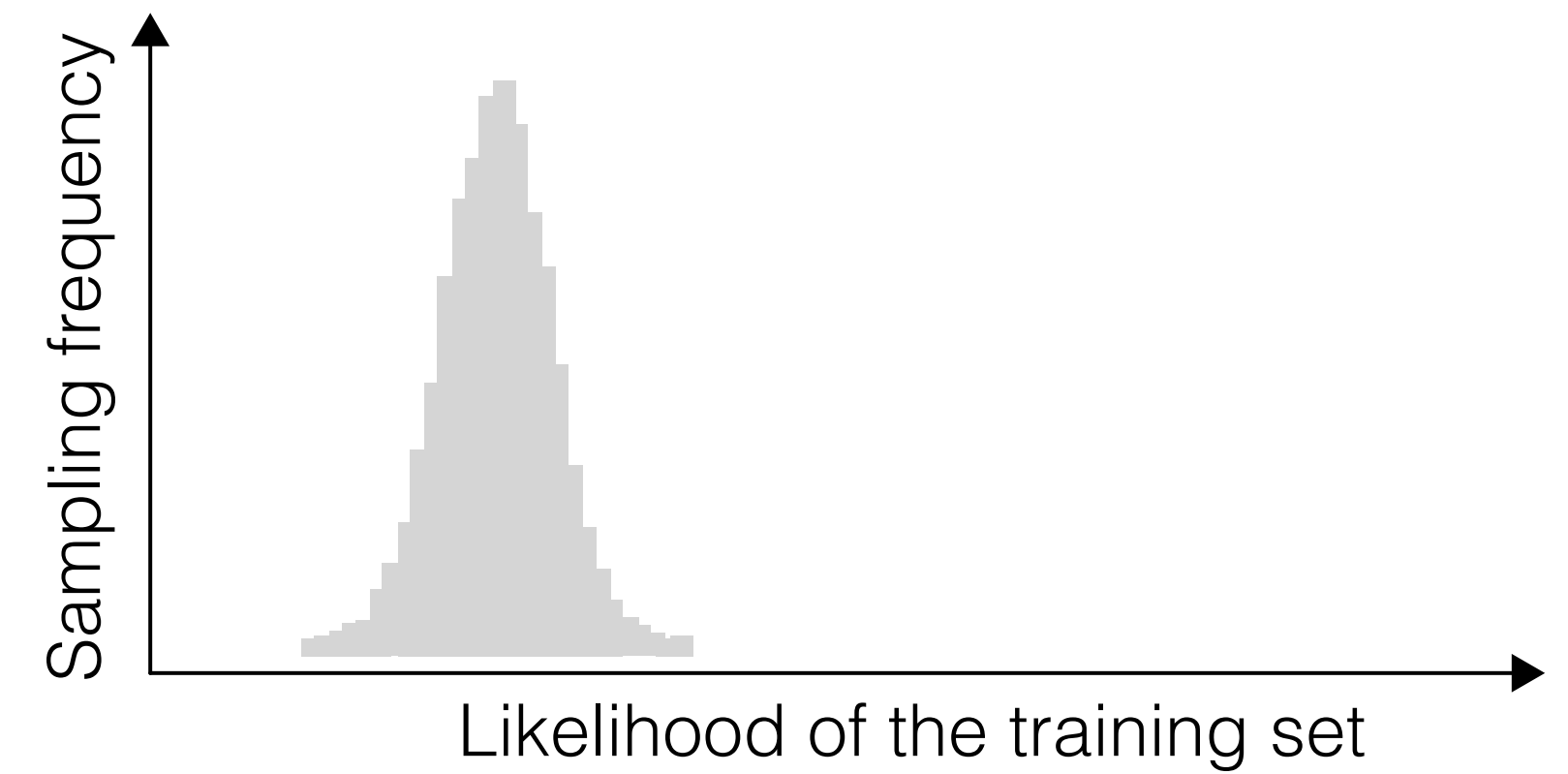
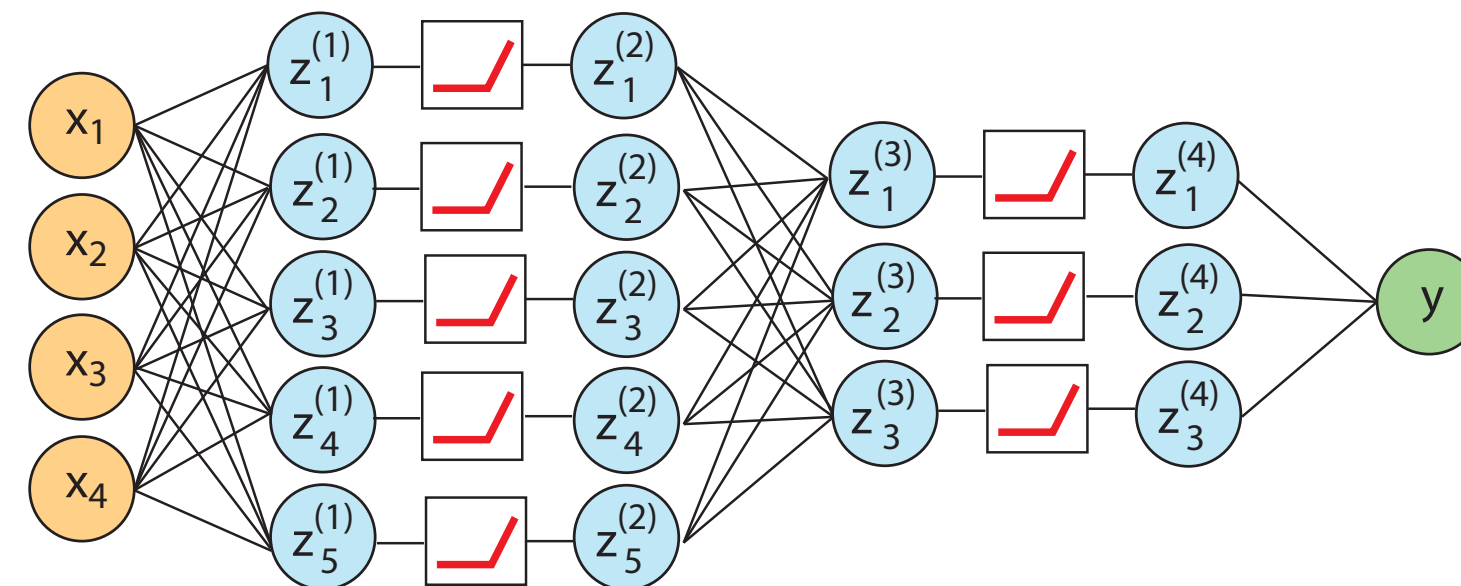
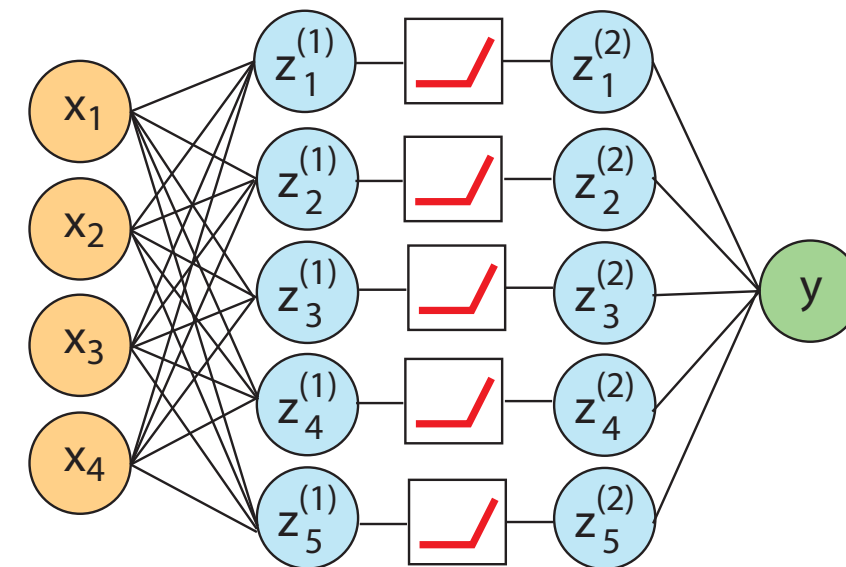
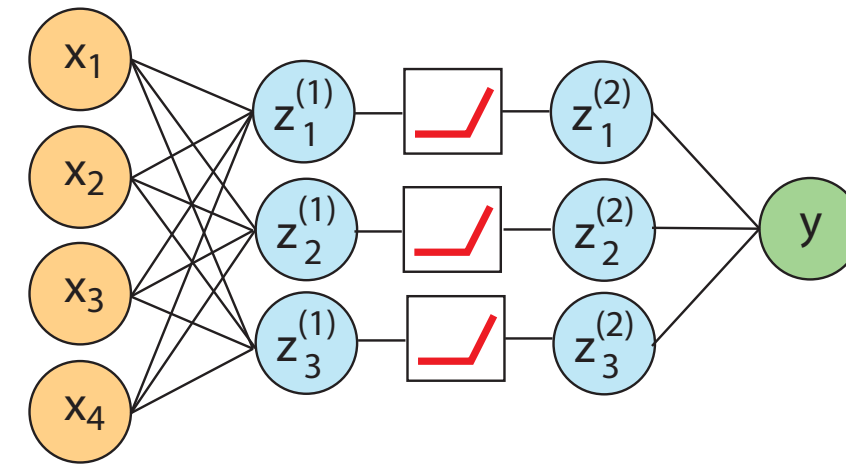


Parameterization of a neural network: how to choose?

Training set

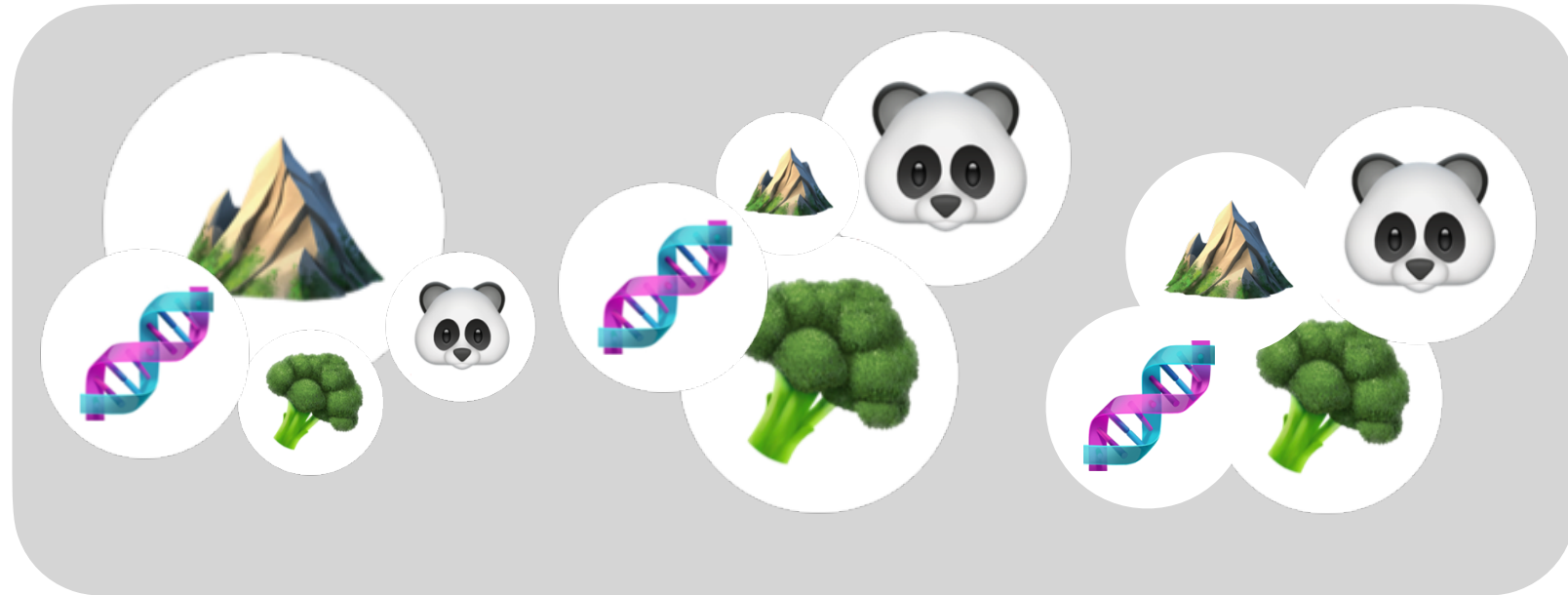


Validation set

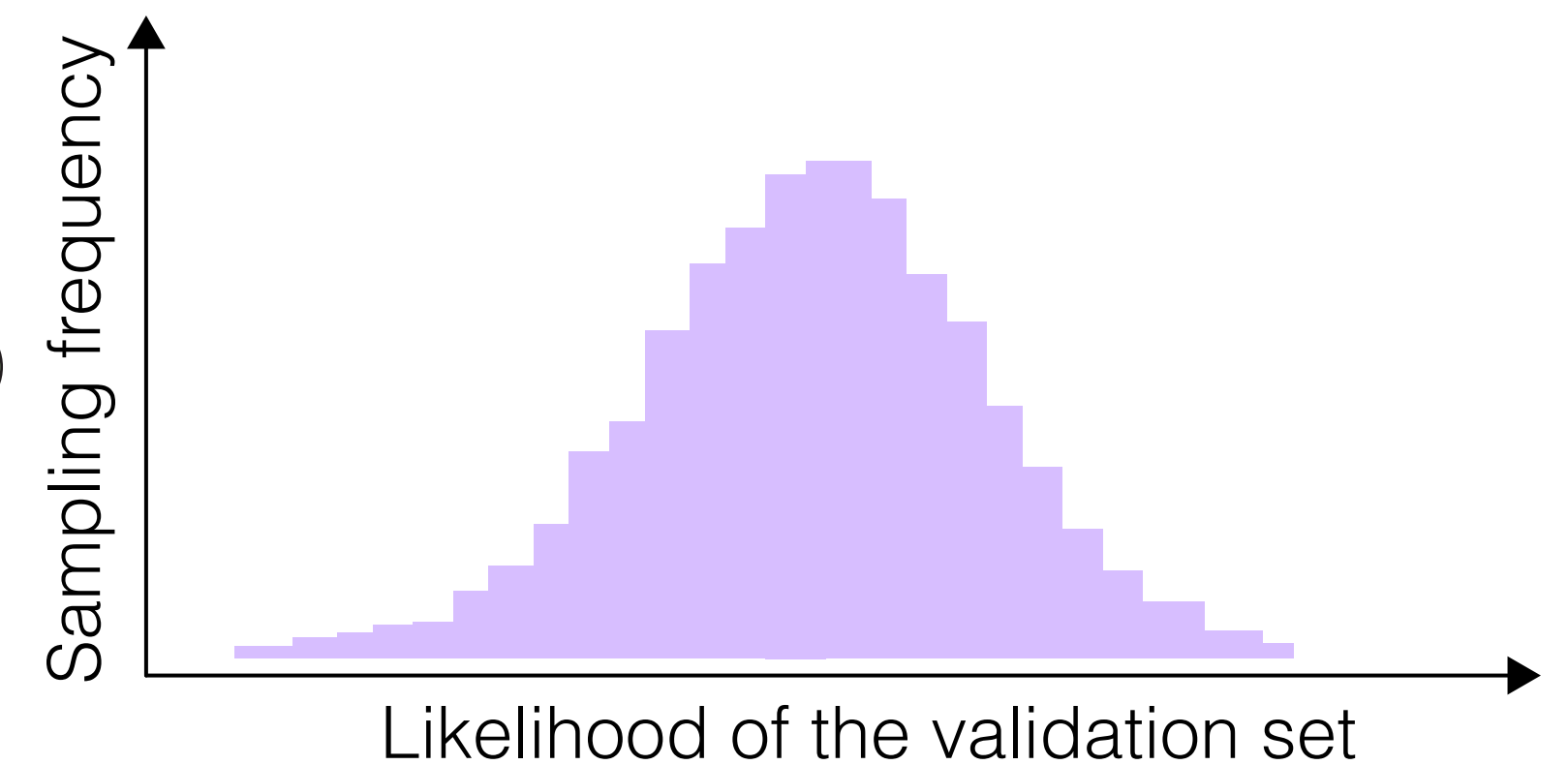
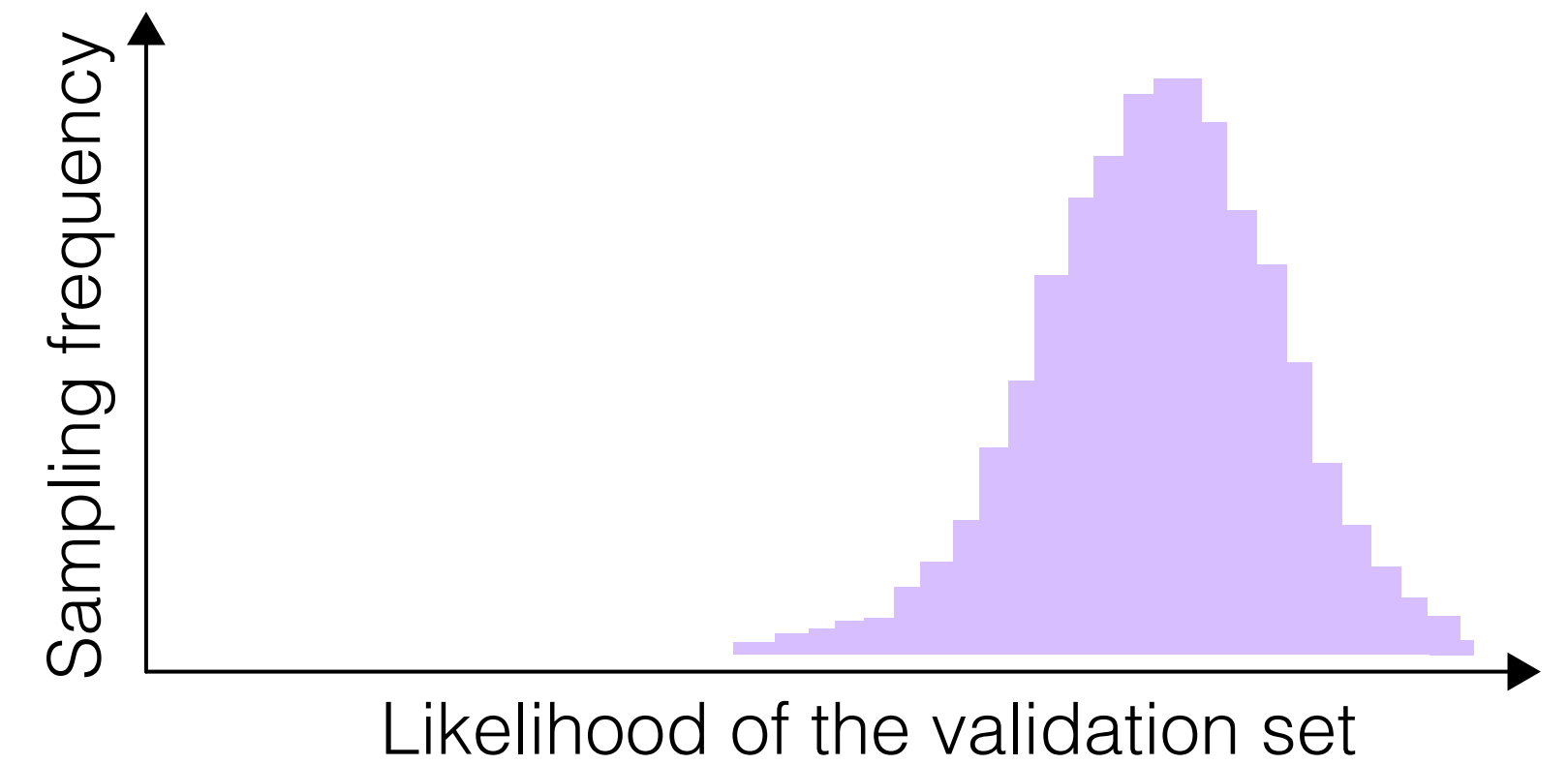
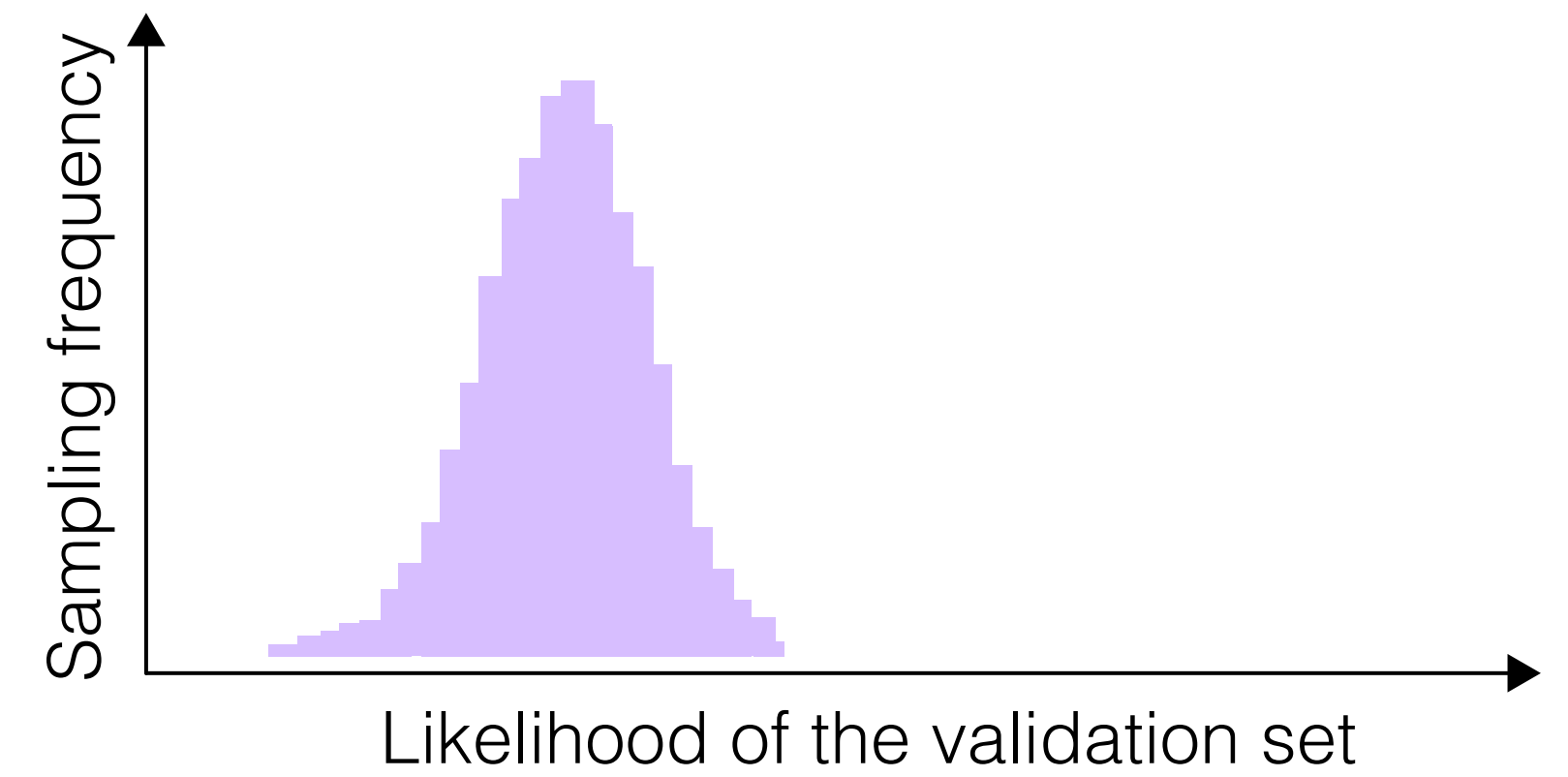
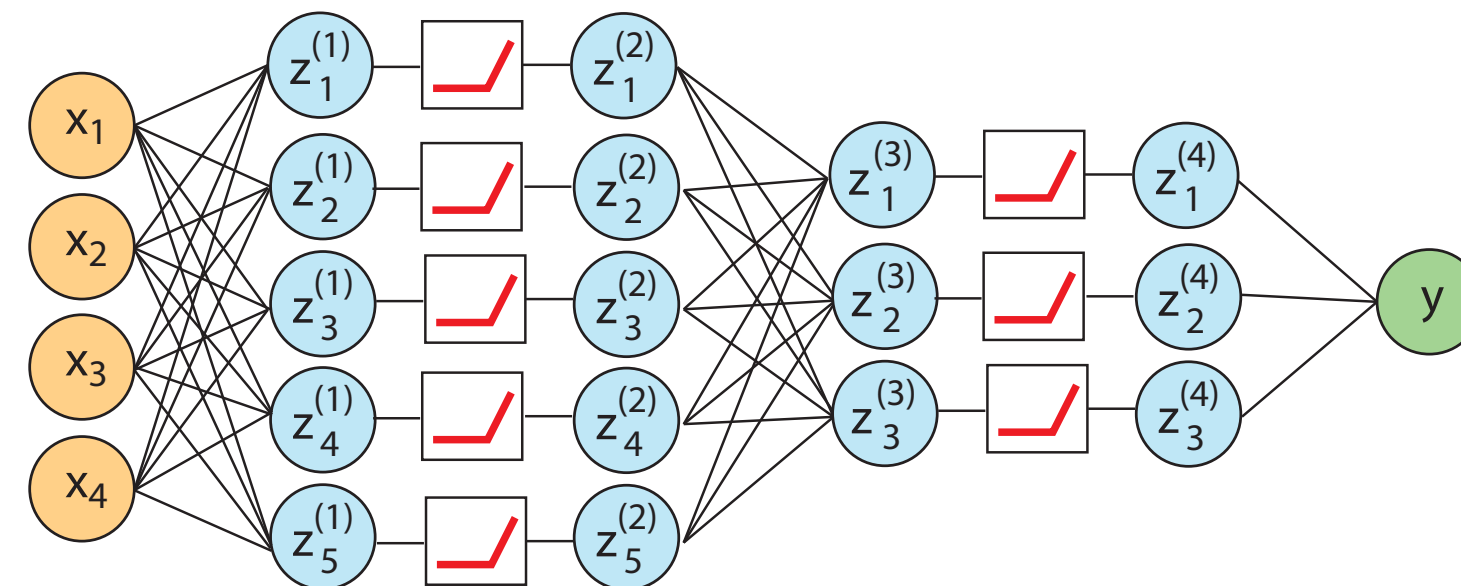
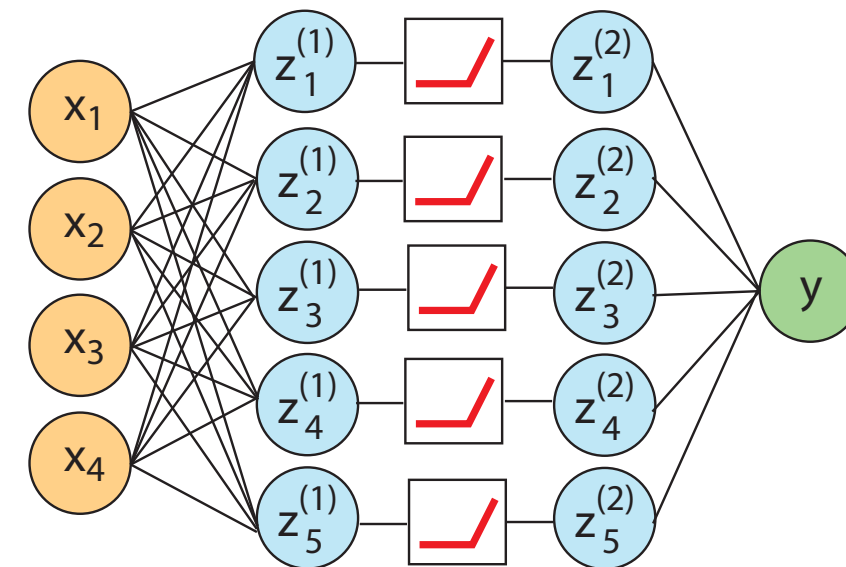
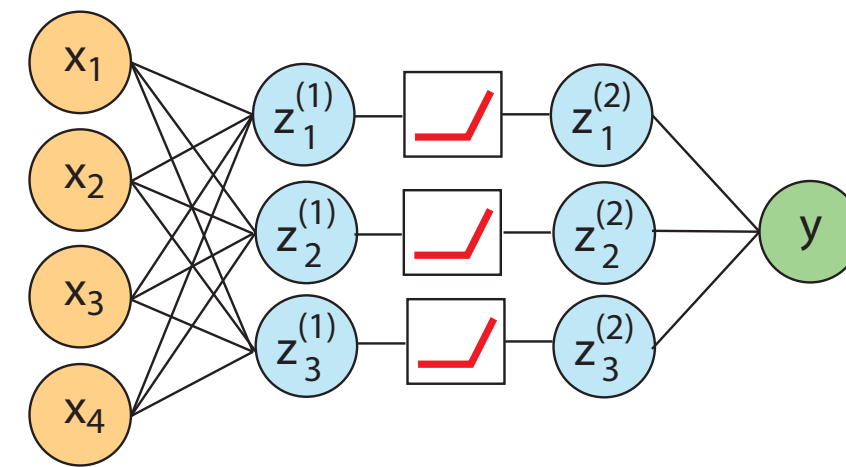


Parameterization of a neural network: how to choose?

Training set



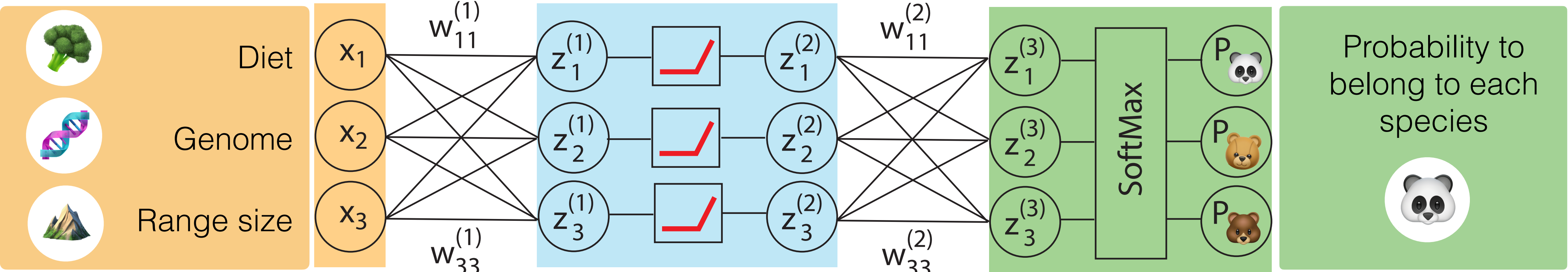
Validation set



Neural networks for classification



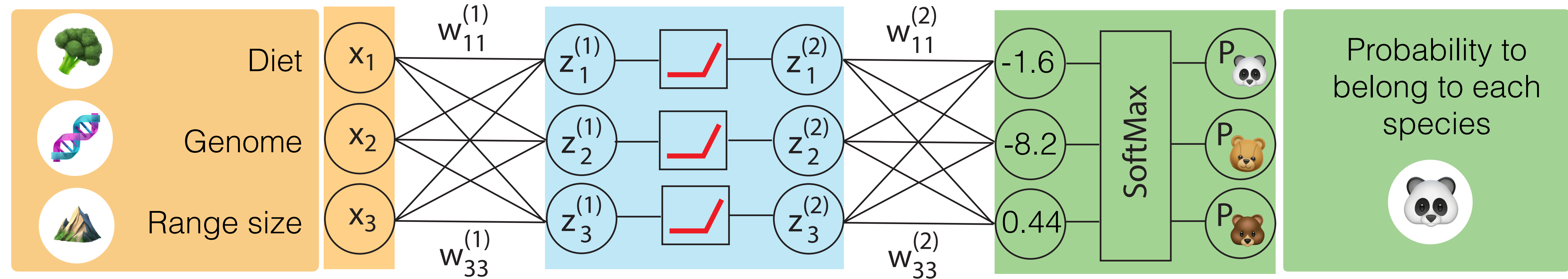
Neural networks for classification



Classification task

Find the parameter for any combination of diet, genome and range size assign the highest probability to the correct species

Neural networks for classification



SoftMax function

Convert arbitrary real values into probabilities

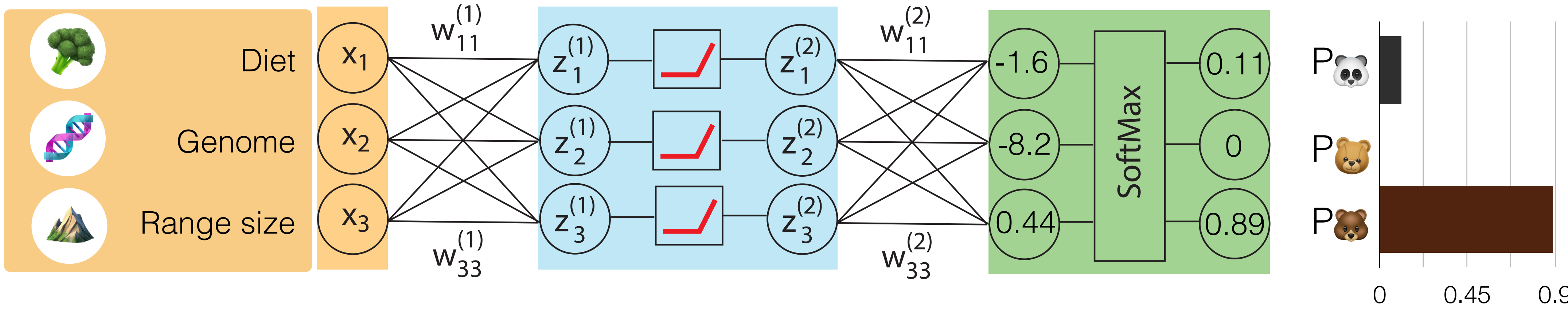
In Python:

```
y = np.exp(z3) / np.sum(np.exp(z3))
```

$$\begin{pmatrix} -1.64 \\ -8.18 \\ 0.44 \end{pmatrix}$$

$z^{(3)}$

Neural networks for classification



SoftMax function

Convert arbitrary real values into probabilities

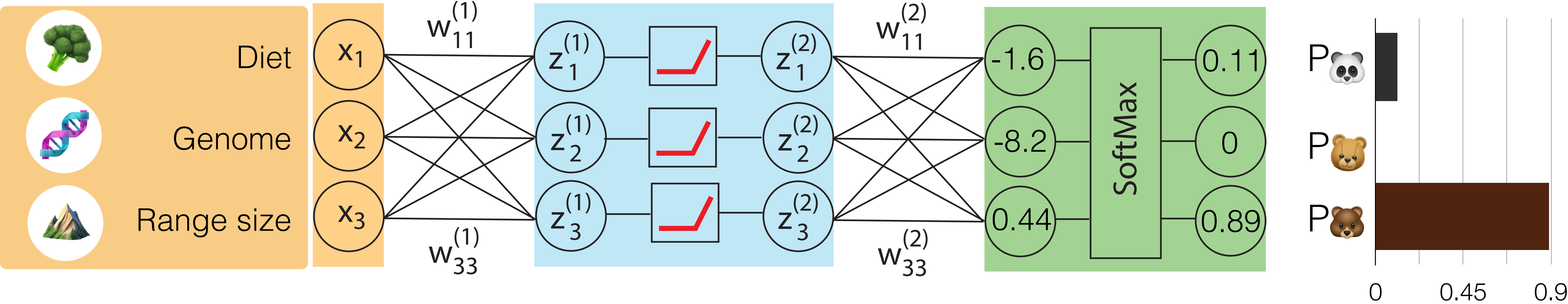
In Python:

```
y = np.exp(z3) / np.sum(np.exp(z3))
```

$$\begin{pmatrix} -1.64 \\ -8.18 \\ 0.44 \end{pmatrix} \begin{matrix} \rightarrow \frac{e^{-1.64}}{(e^{-1.64} + e^{-8.18} + e^{0.44})} \\ \rightarrow \frac{e^{-8.18}}{(e^{-1.64} + e^{-8.18} + e^{0.44})} \\ \rightarrow \frac{e^{0.44}}{(e^{-1.64} + e^{-8.18} + e^{0.44})} \end{matrix} = \begin{pmatrix} 0.11 \\ 0 \\ 0.89 \end{pmatrix}$$

SoftMax

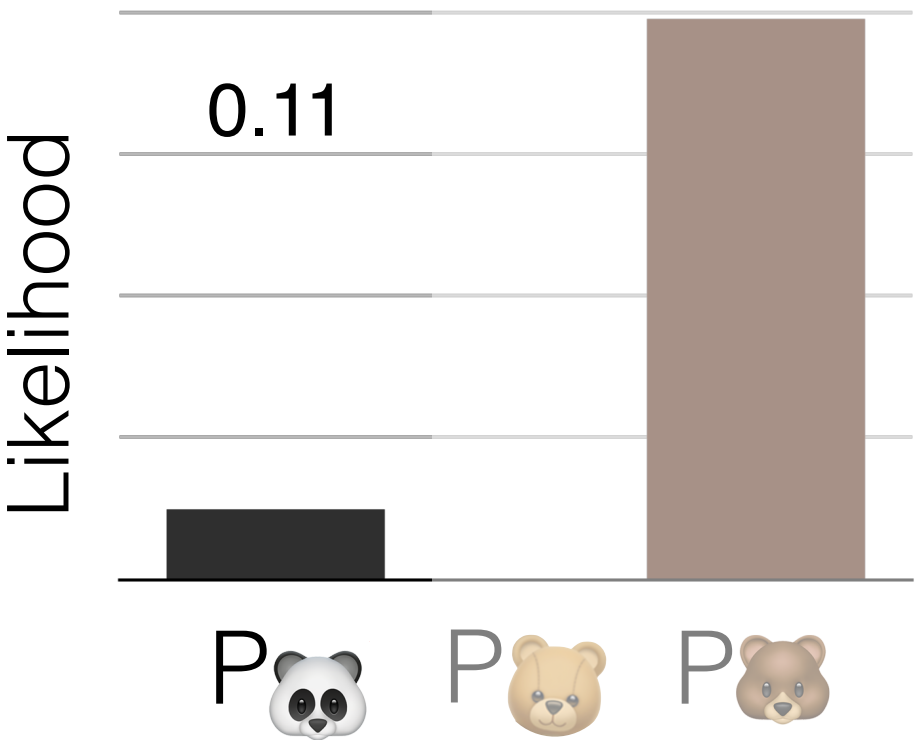
Neural networks for classification



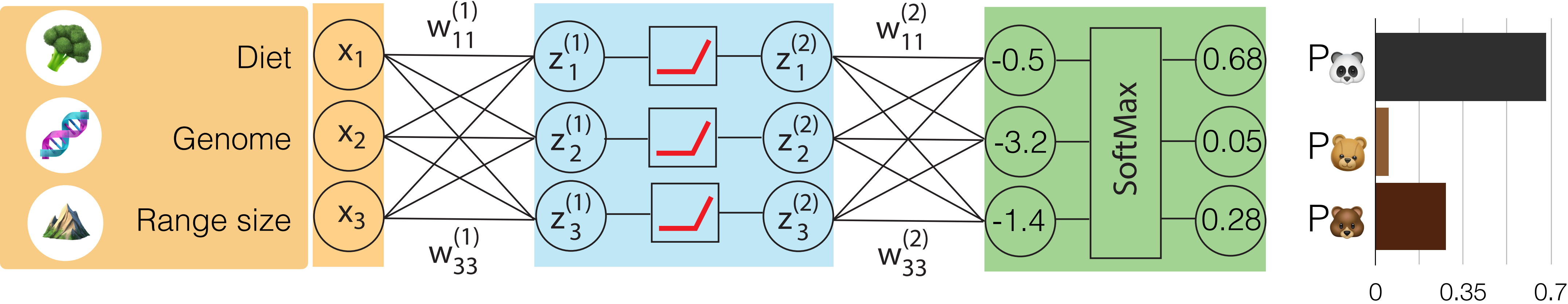
Likelihood function in a classification model



Probability mass function of a categorical distribution



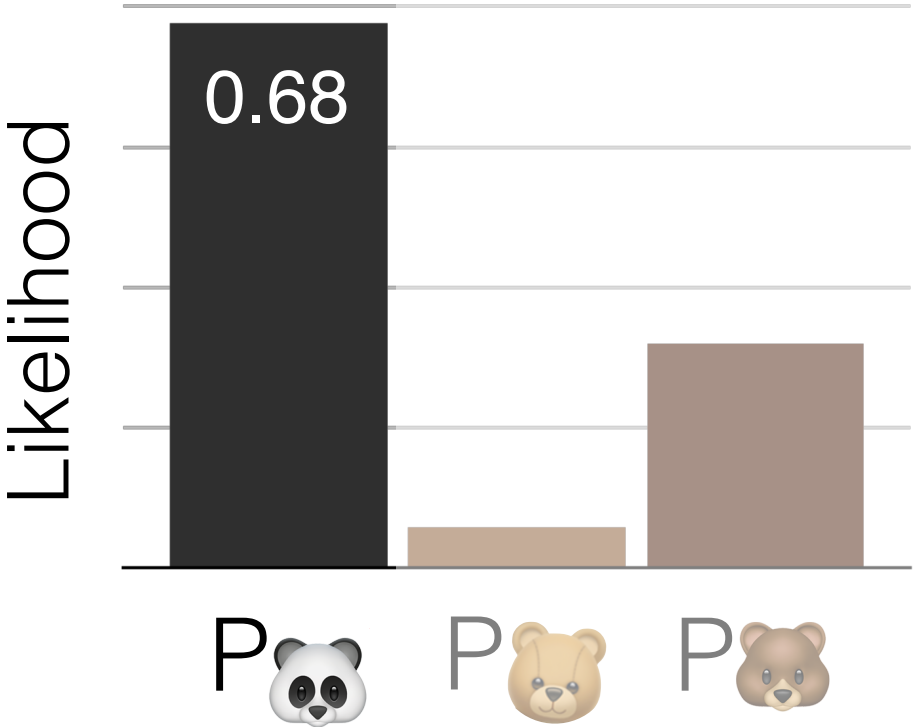
Neural networks for classification



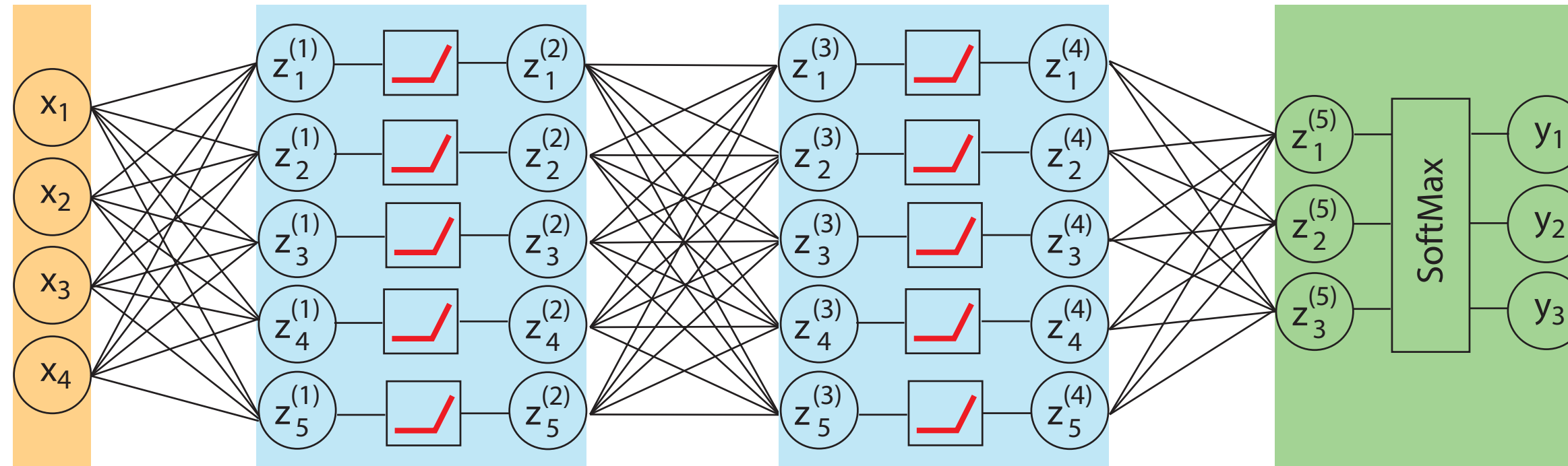
Likelihood function in a classification model



Probability mass function of a categorical distribution



Likelihood in a neural network model of classification

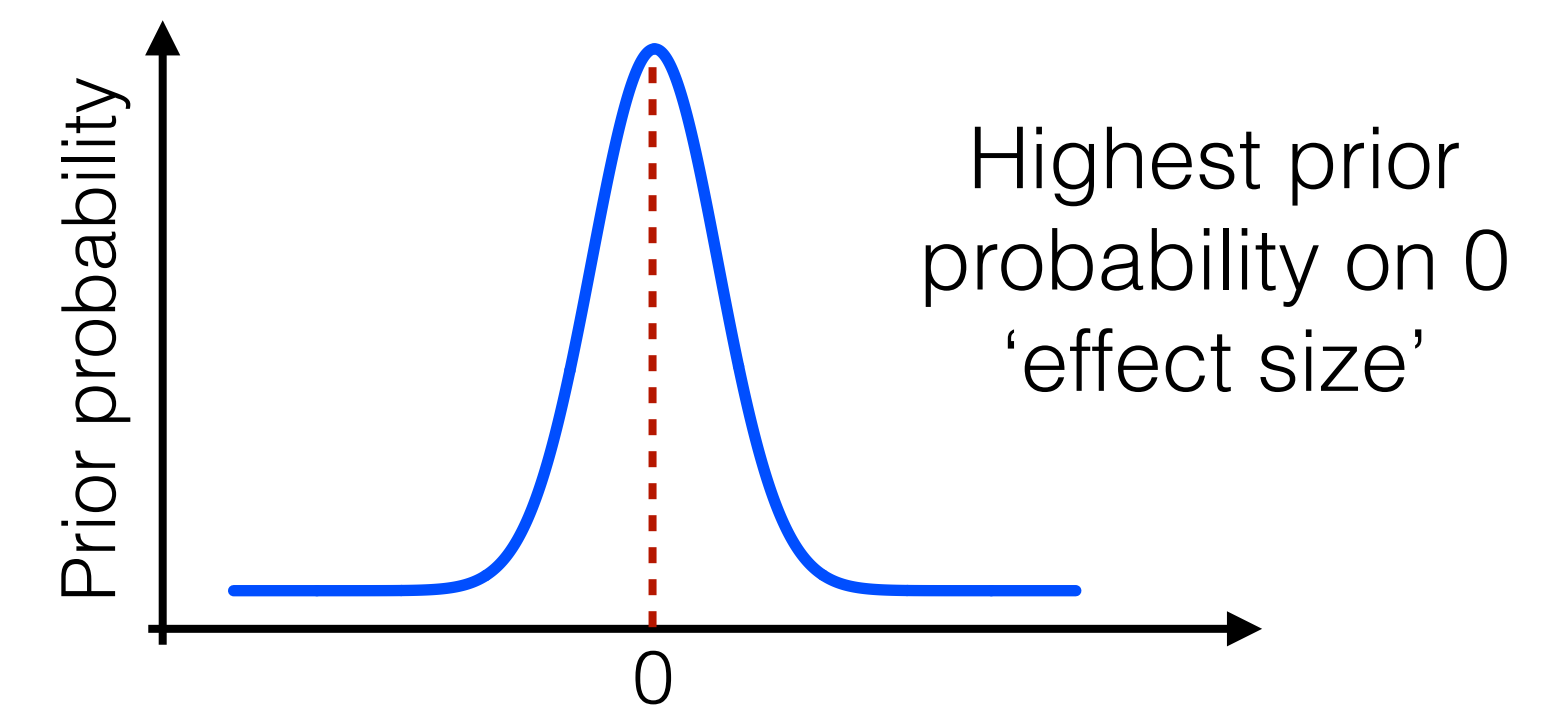
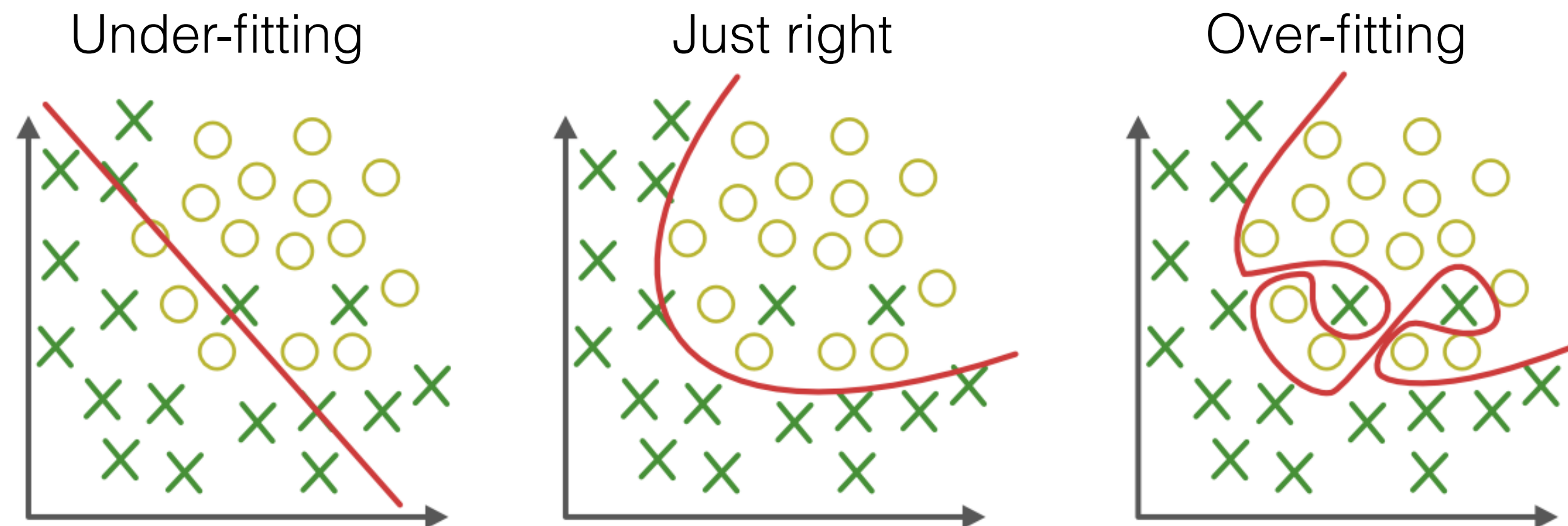


A maximum likelihood optimization of an NN would inevitably lead to over-fitting because NNs are over-parameterized

In a BNN the priors have a regularizing effect that limits the risk of overfitting

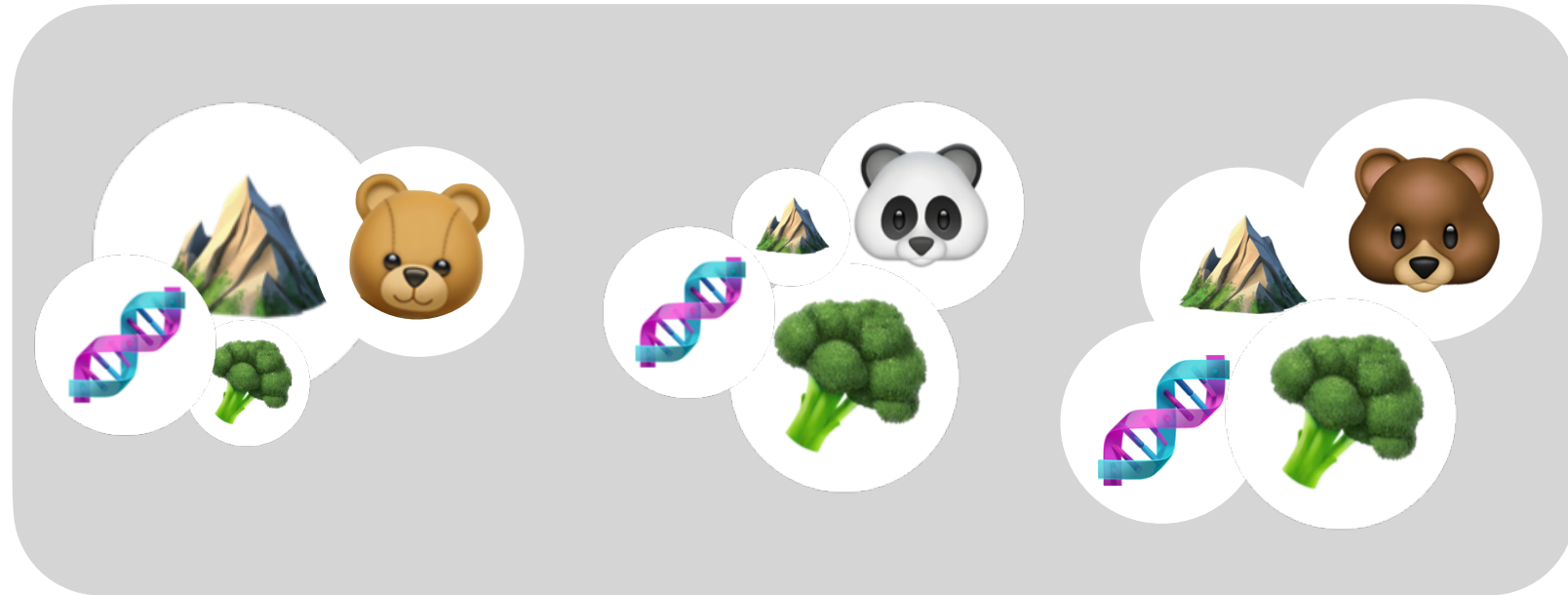
Priors on the weights

$$P(w) \sim \mathcal{N}(0, 1)$$

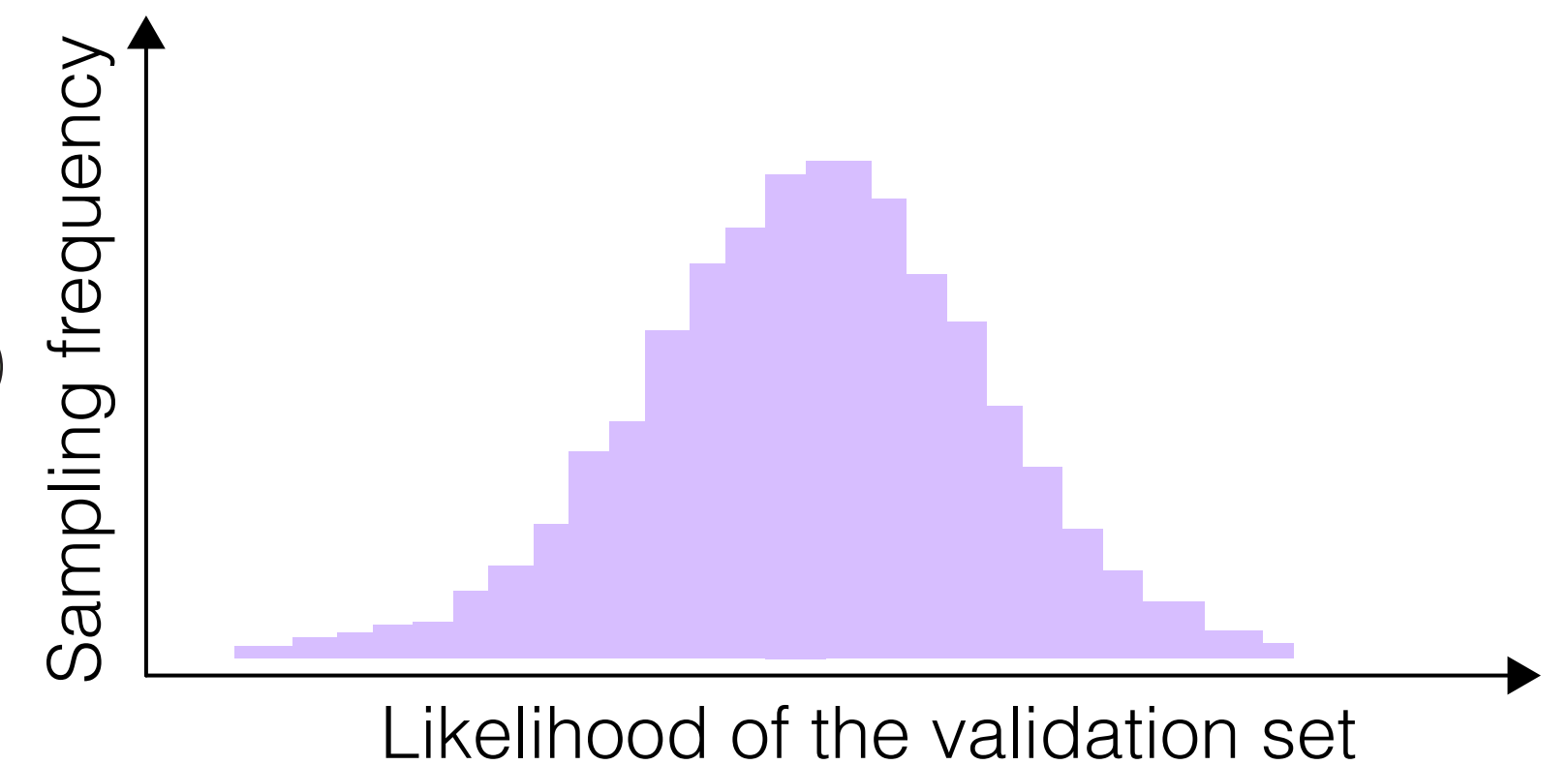
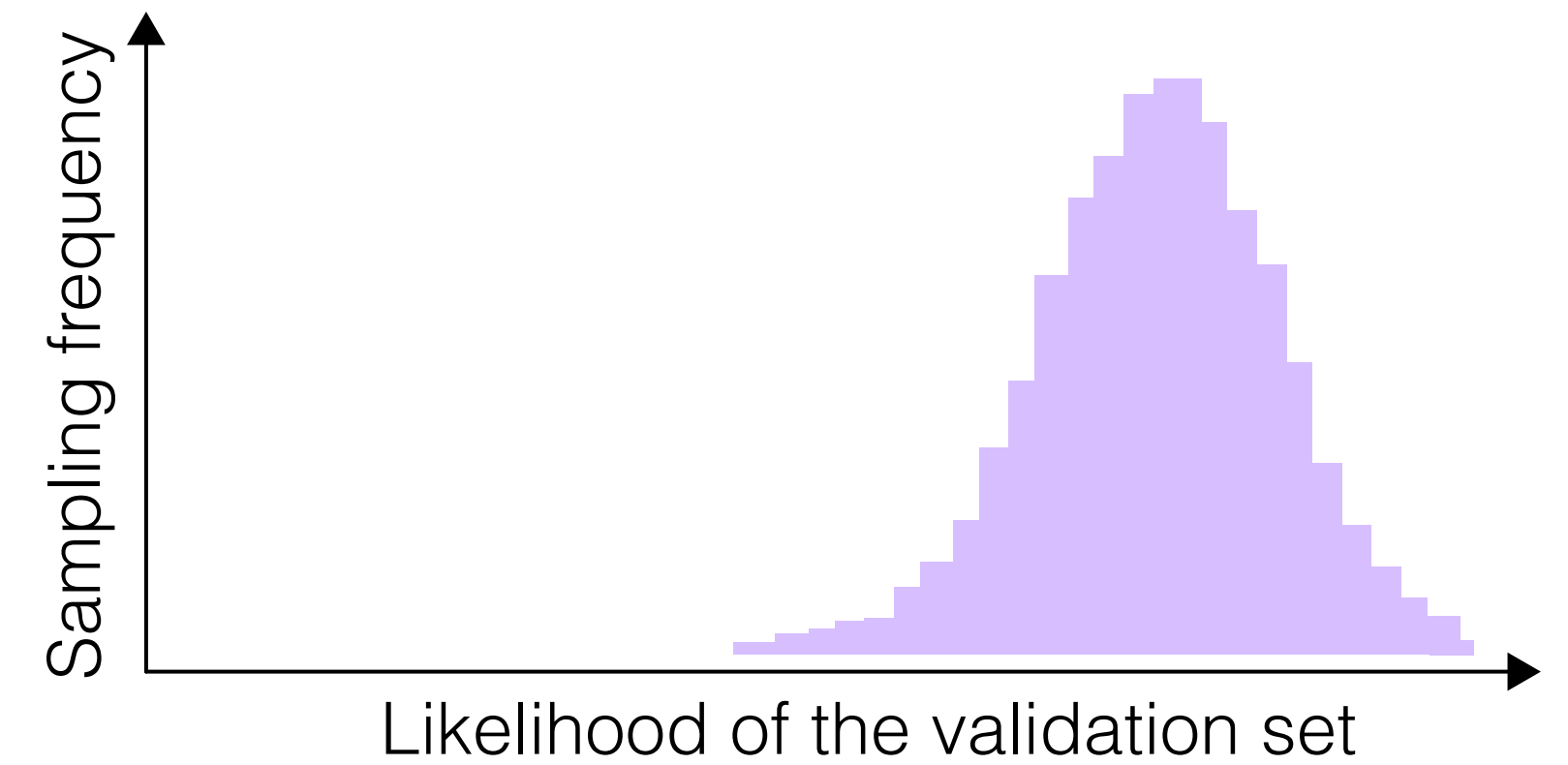
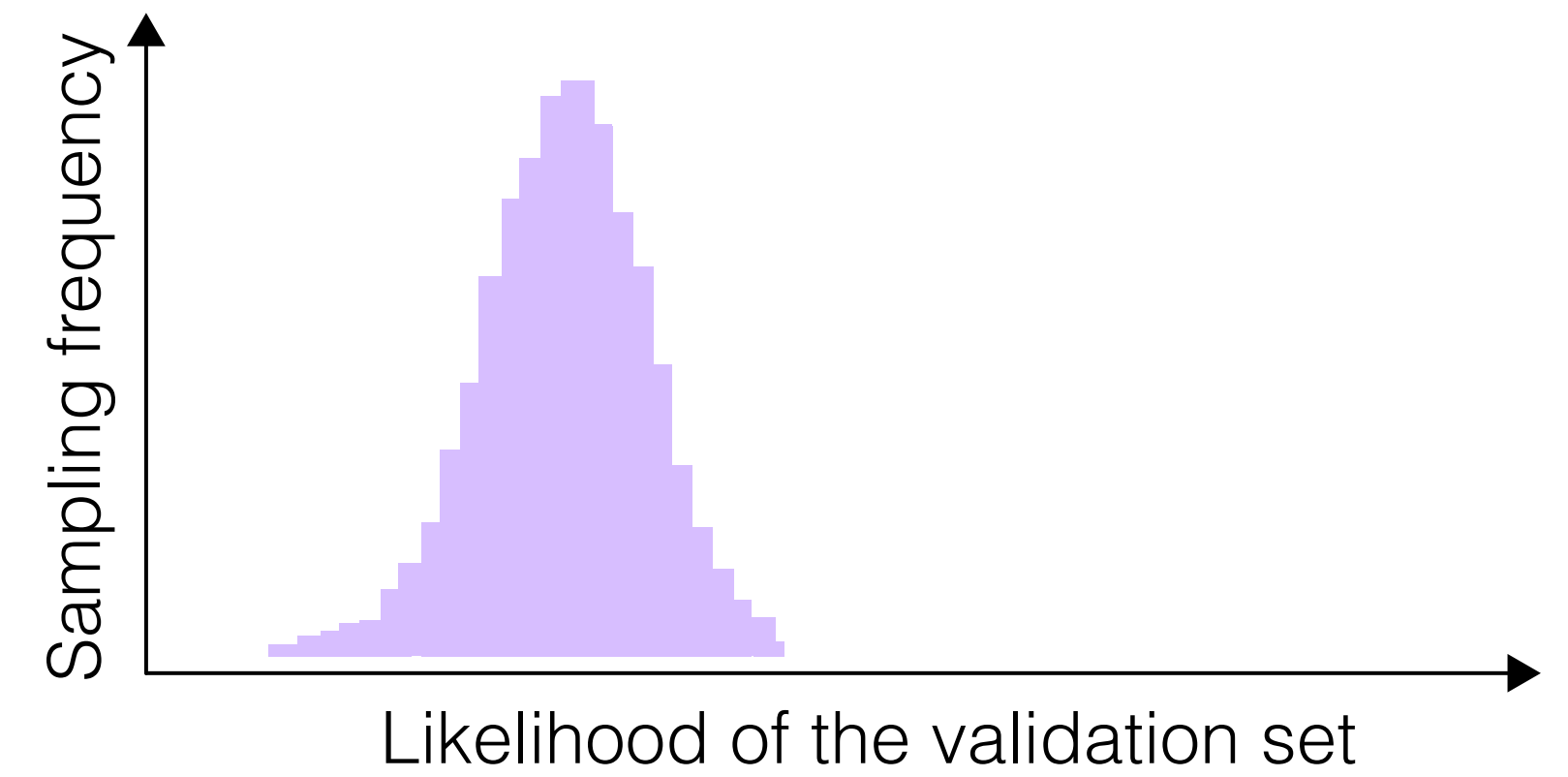
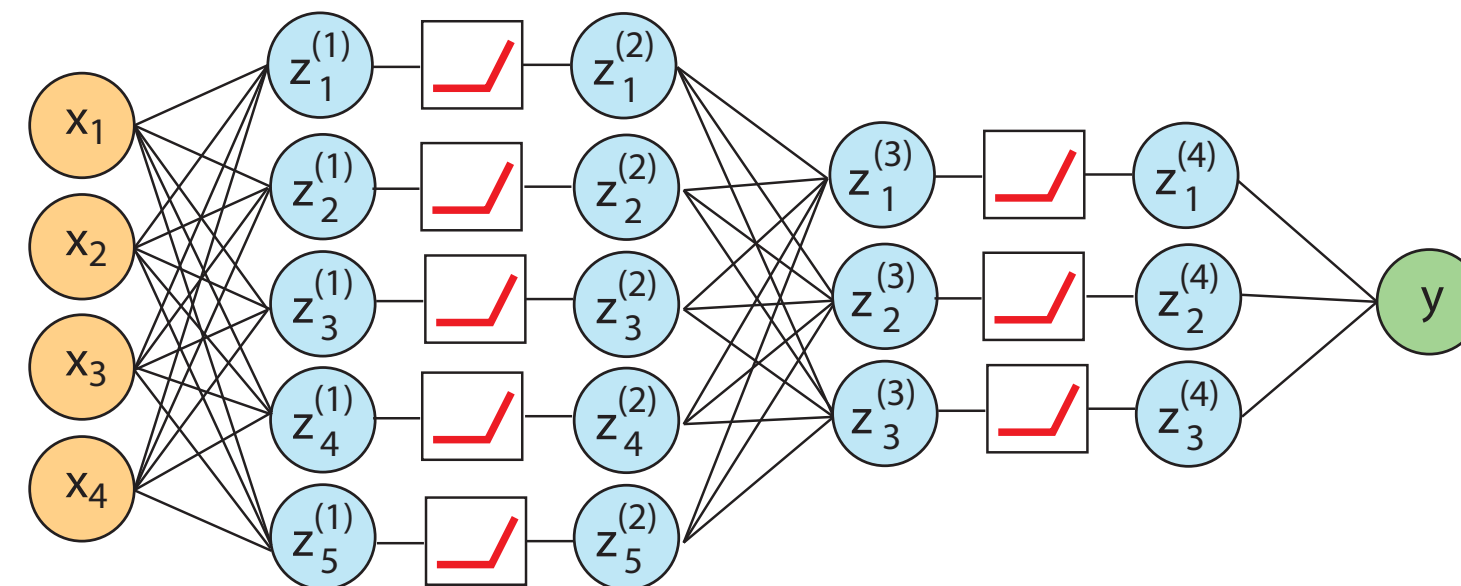
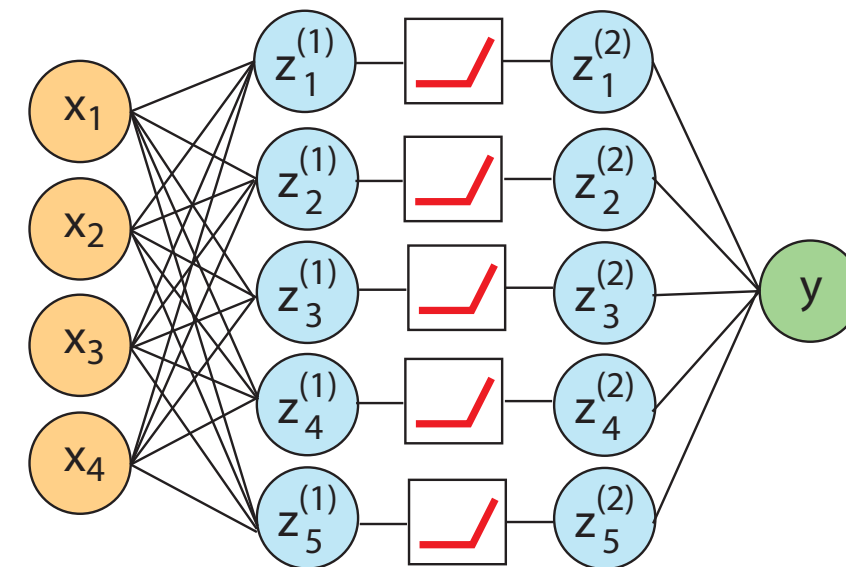
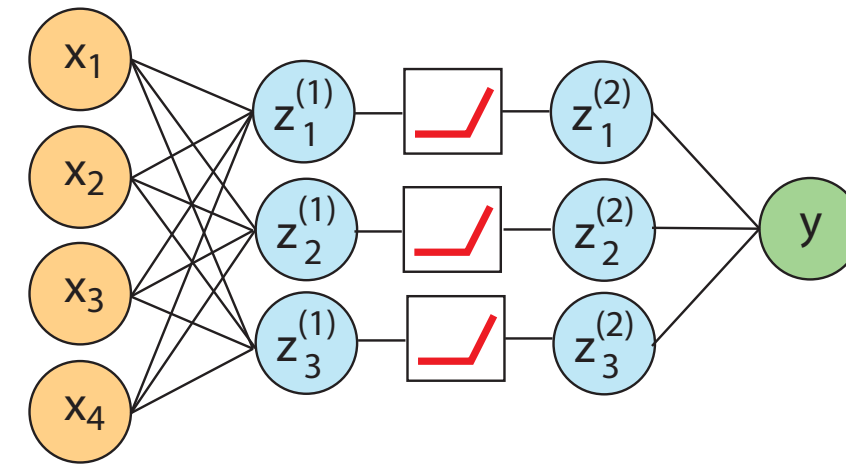


Parameterization of a neural network: how to choose?

Training set

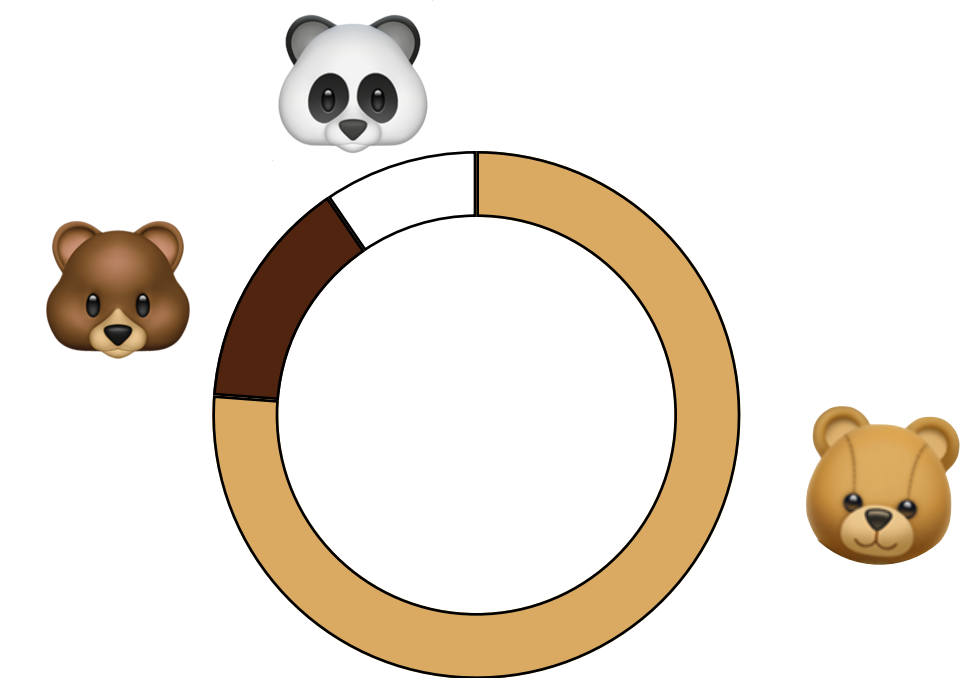
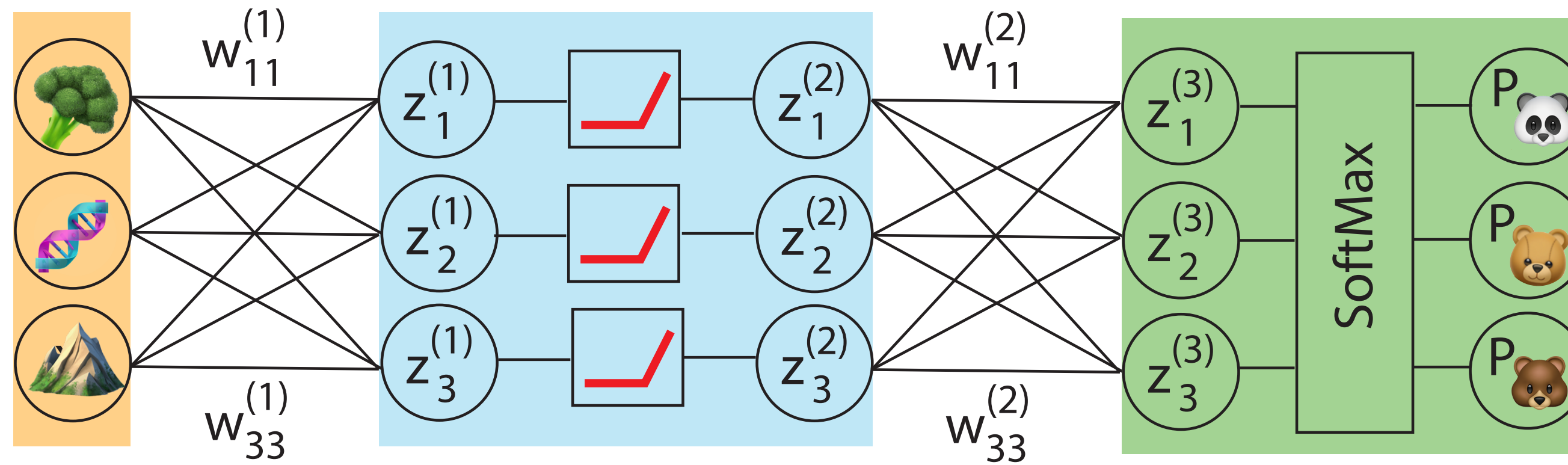
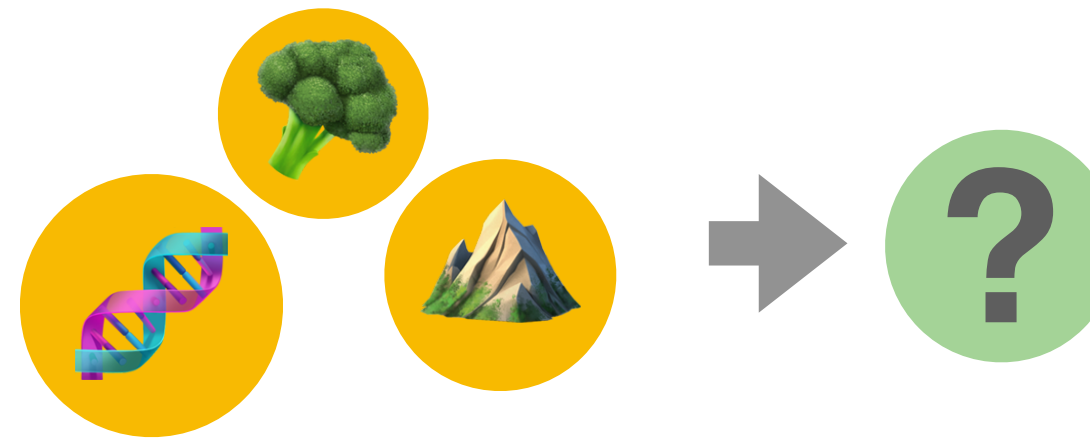


Validation set



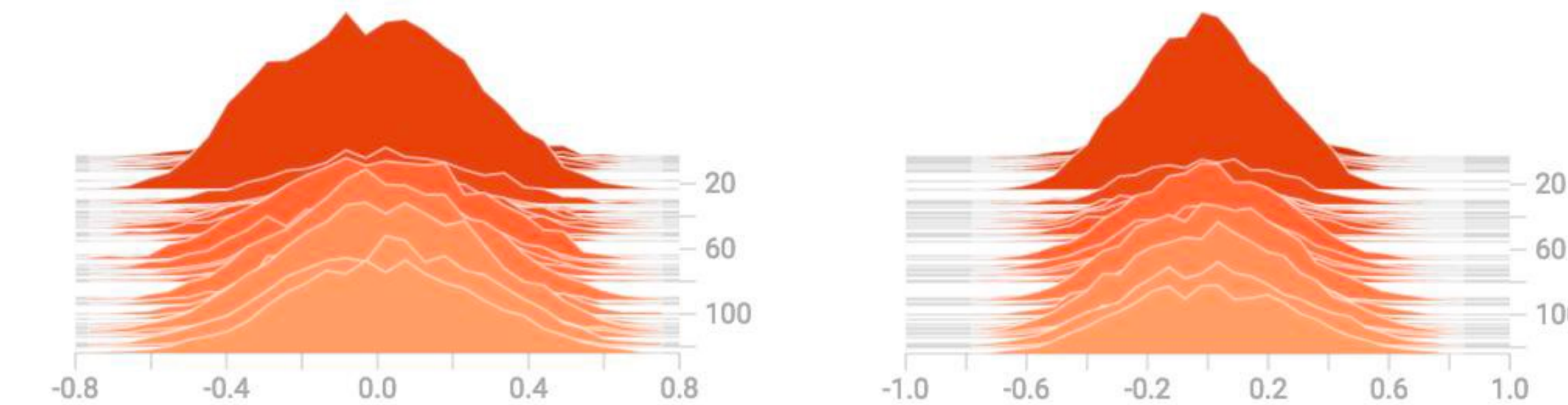
Predictions using a trained BNN

Predicting body mass from
diet, genome, range size



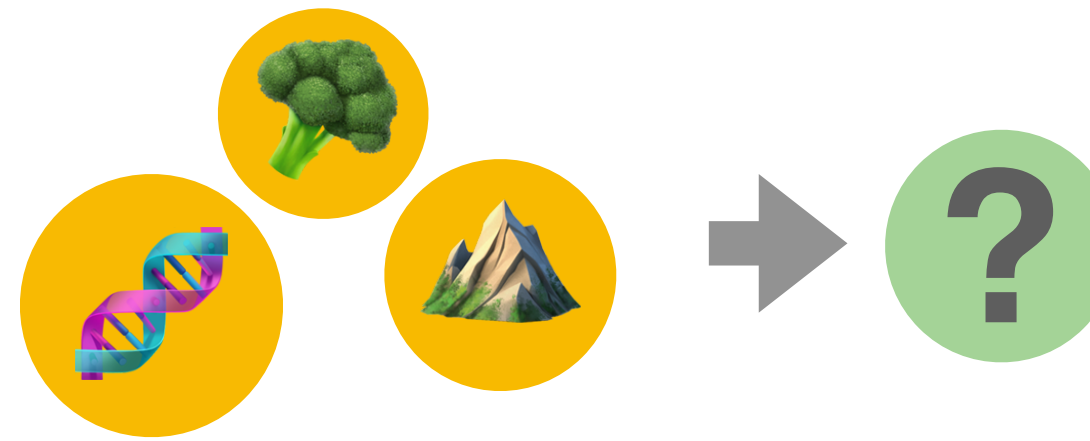
**Posterior probability
associated to each class**

Posterior samples
of the weights

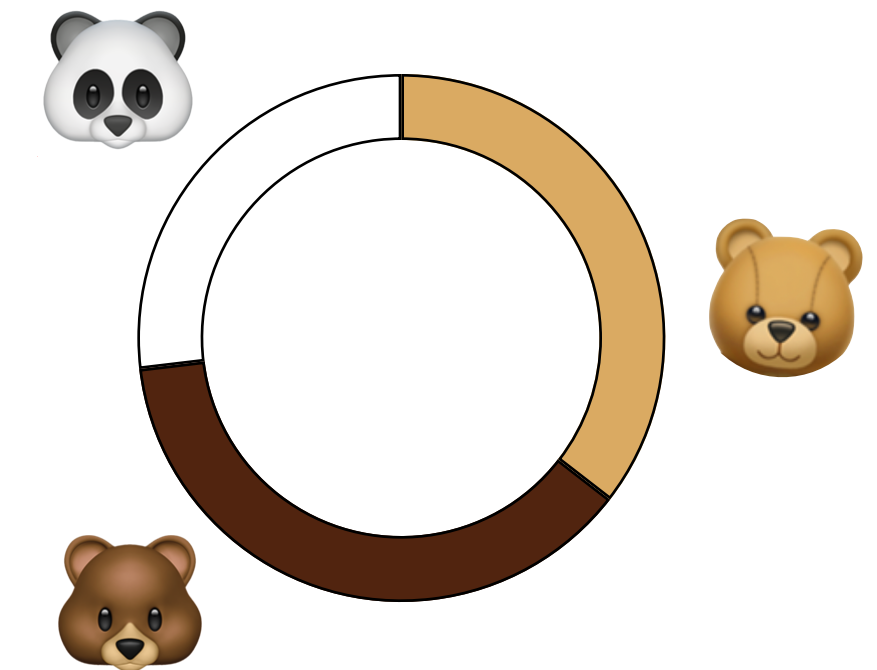
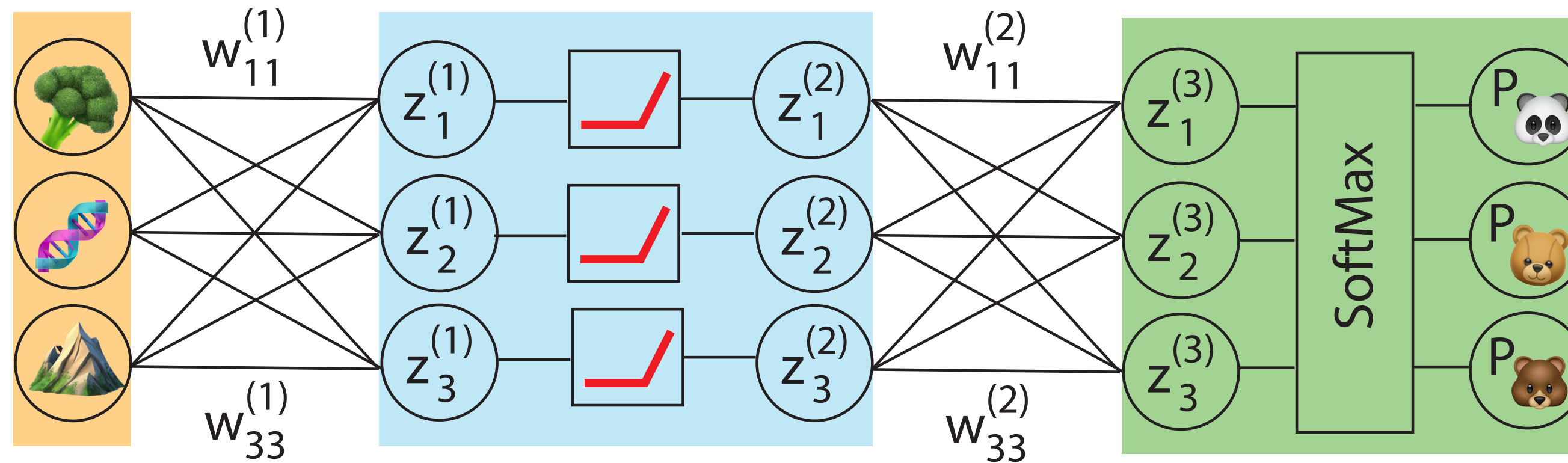


Predictions using a trained BNN – identifying unknowns

Predicting body mass from
diet, genome, range size

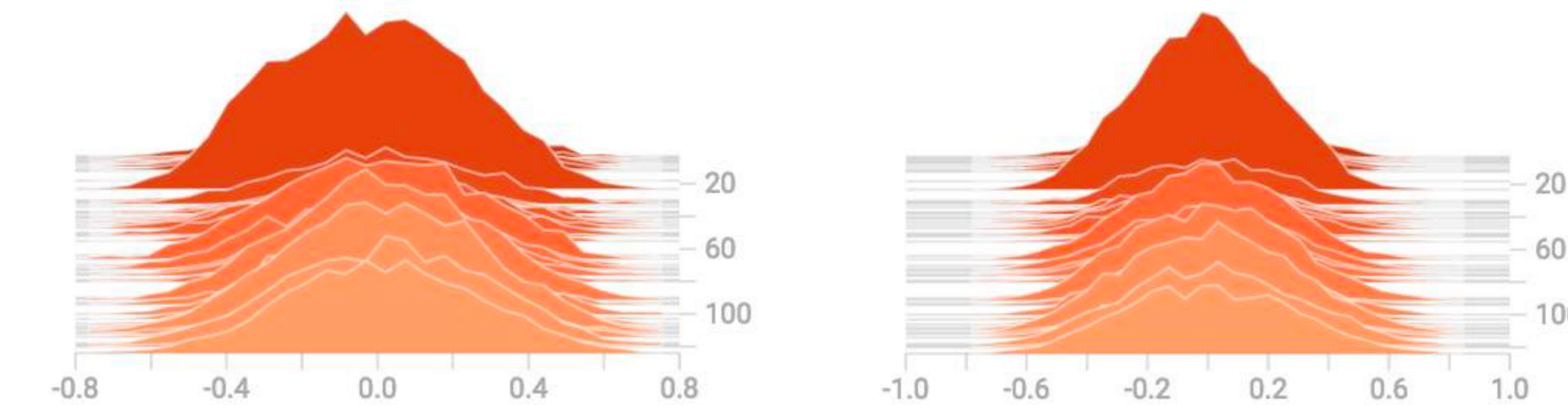


 Ground truth:
out-of-distribution!



**Low posterior probabilities
among classes**

Posterior samples
of the weights



Model performance: classification accuracy

Accuracy: $\frac{33}{37} = 89\%$

Error rate: $\frac{4}{37} = 11\%$



Confusion matrix:

		Predicted		
True		14	0	0
		0	9	1
		1	2	10

Final evaluation of the model performance: test accuracy

Accuracy: $\frac{46}{50} = 92\%$

Error rate: $\frac{4}{50} = 8\%$

Confusion matrix:

		Predicted		
				
True		46	0	0
		1	0	0
		3	0	0

Maybe this model will classify *everything* as a panda



Careful with imbalanced classes in the dataset!

The NN is a blackbox model



The parameterization of a NN is arbitrary and does not aim or attempt to reflect a process or a specific distribution

The weights are not directly interpretable parameters and the main parameters of interest are in the output layer

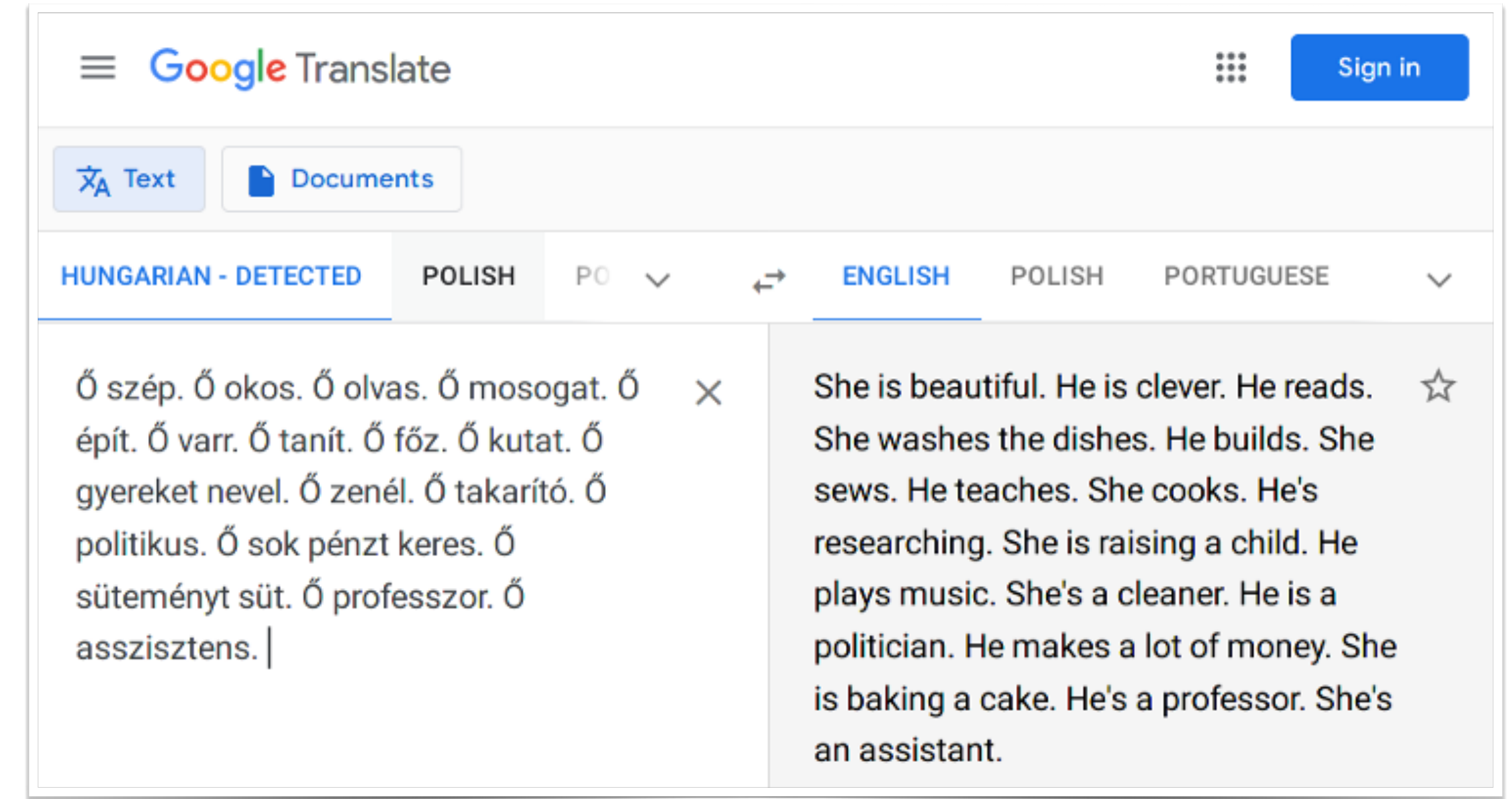


Check out our Bayesian neural network implementation!

<https://github.com/dsilvestro/npBNN>

Silvestro and Andermann 2020, arXiv: 2005.04987v1

Problems and misuses in AI



The Observer
Artificial intelligence (AI)

AI poses national security threat, warns terror watchdog

Security services fear the new technology could be used to groom vulnerable people

Mark Townsend *Home Affairs Editor*
@townsendmark
Sun 4 Jun 2023 07:00 BST

